Instructional Influences on Content Area Explanations and Representational Knowledge: Evidence for the Construct Validity of Measures of Principled Understanding

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ABSTRACT

Recent calls for understanding-based mathematics instruction imply a need for alternative kinds of assessment (National Council of Teachers of Mathematics, 1989; Webb & Romberg, 1992; Wilson, 1992). One of the most significant implications of the National Council of Teachers of Mathematics *Standards* and other reform documents is that assessment strategies should be aligned with theoretical analyses of the construct domains they are intended to assess. To be useful for instructional purposes, assessments should also be sensitive to changes in students' understanding. At present there are few published reports of assessment or instruction strategies based on systematic models of mathematics knowledge, and even fewer reports on the construct validation of such strategies.

The purpose of this study was to develop and test measures of principled understanding, based on conceptual analysis of a key domain in elementary mathematics, fractions. The study compared two differently-instructed groups of fifth-grade students. One group, comprising 11 randomly-assigned classrooms, received explicit instruction on fraction principles deriving from measurement situations. Another group of 11 randomly-assigned classrooms practiced using more traditional area representations and computation techniques. Before instruction all students took a 60-item measure of general fraction knowledge. After instruction both groups completed representational knowledge and explanation tasks, in addition to tests of problem solving, conceptual knowledge, and computation knowledge. The problem-solving, conceptual knowledge, and computation tests, in addition to teacher ratings and standardized achievement test scores, were used to validate the representational knowledge and explanation measures.

One of the most important findings is that students who received explicit instruction on fraction principles performed better than students in the other group on nearly all measures of principled understanding, and equally well on measures of computation. In response to instruction, many students appeared to add new ideas about fractions to their existing repertoires, without discarding or reworking previously learned ideas, and to draw upon all available ideas in constructing their explanations. Results also showed the feasibility of assessing important aspects of mathematical understanding through students' use of mathematical representations and language. It was found that mathematics justifications and explanations could be reliably scored and that students used their representational knowledge when they constructed justifications and explanations.

INSTRUCTIONAL INFLUENCES ON CONTENT AREA EXPLANATIONS AND REPRESENTATIONAL KNOWLEDGE: EVIDENCE FOR THE CONSTRUCT VALIDITY OF MEASURES OF PRINCIPLED UNDERSTANDING–MATHEMATICS

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INTRODUCTION

Statement of the Problem

In response to unacceptable levels of mathematics achievement in the U.S., and to the problematic nature of prevailing instruction and assessment practice, a number of influential professional groups have put forth compelling proposals for the reform of mathematics education (e.g., California Board of Education, 1991; National Council of Supervisors of Mathematics, 1989; National Council of Teachers of Mathematics [NCTM], 1989; National Research Council, 1989). These proposals express a new conception of mathematics achievement, in which understanding plays a central role and mathematical knowledge is conceived as a system of knowledge about mathematical concepts, operations, symbols,[1] and

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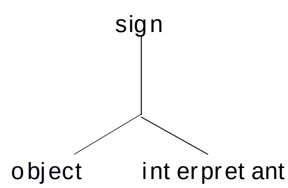
[1] Following usage common in mathematics education and psychology literature, the term "symbol" is used here to denote an element of what has been called mathematical "orthography," such as 2, +, =, or 3/5, whose referent is typically a mathematical concept or operation. In Peircean semiotic theory (e.g., Houser, 1987; Peirce, 1932), however, notational elements are not considered to be symbols in themselves; instead, a symbol consists of the following set of relations:



The Semiotic Triangle

In Peircean semiotics, the meaning of a symbol inheres in its triadic structure, not in a mechanical mapping between symbol and object or between symbol and interpretant. According to Peirce, the elements in this relational network cannot be decomposed without altering the meaning. Symbols, concepts and operations, and situations—the elements of mathematical knowledge assessed in this study—are respectively analogous to the elements of Peirce's triad: sign, interpretant, object. Greeno expresses a similar interpretation of mathematical meaning: "Processes of learning to construct and reason with numerical mental models probably require coordination of symbolic representations of numbers with physical quantities or mental models situations (e.g., Cole, 1986, 1990; Webb & Romberg, 1992; Wilson, 1992). This conception contrasts with the more common "instrumental" view, in which mathematics is implicitly or explicitly defined as a collection of memorized "problem-solving" procedures that have little or no relation to one another (Skemp, 1976). Unfortunately, mathematics as a system of knowledge is not an idea that has taken hold in mathematics education: The instrumental view dominates conventional practice.

In the typical elementary school classroom, 85% or more of the available instruction time is devoted to demonstrating and practicing computational procedures; 15% or less is given to the development of conceptual understanding (Peterson & Fennema, 1985; Porter, 1989; Romberg & Carpenter, 1986). Most procedures are taught and practiced in isolation from one another and from any conceptual knowledge students may have (Davis & McKnight, 1980; Fey, 1979; McKnight et al., 1987; Porter, 1989; Romberg, 1987; Skemp, 1976; Stodolsky, 1988). Learners are supposed to acquire the procedures, which are dignified by mathematical names such as "addition" and "subtraction" (McLellan & Dewey, 1895), by extended repetitious practice, usually after watching demonstrations on a blackboard. This rote-acquisition-of-algorithms method has persisted for at least a century and is not limited to the United States: researchers in the Second International Study of Mathematics found that current mathematics teaching can be categorized almost universally as formal lectures on procedures and rules for manipulating symbols, followed by lengthy rehearsal (McKnight, 1987).

Rote-acquisition approaches to instruction imply that mathematics is an unstructured collection of meaningless procedures, a position contradicted by the findings of cognitive scientists and other researchers studying the psychology of mathematics (e.g., Schoenfeld, 1987), and by a massive body of evidence suggesting that rote learning of isolated rules leads to mathematical incompetence. In general students do not understand the mathematics they have been "taught" and cannot remember or apply procedures they have practiced for many years (e.g., Brown & VanLehn, 1982; Brownell, 1945; Carpenter et al., 1988; Davis, 1984; Davis & McKnight, 1980; Dossey, Mullis, Lindquist, & Chambers, 1988; Greeno, 1978; Hart, 1981; Hiebert, 1988; Lesh, Landau, &

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of quantities" (1991, p. 197), that is, links between interpretants ("numerical mental models"), signs ("symbolic representations"), and objects ("physical quantities or mental models of quantities").

Hamilton, 1983; National Council of Teachers of Mathematics, 1989; Resnick, 1987; Romberg & Carpenter, 1986; Silver, 1986; Silver & Carpenter, 1990). Approaches that take mathematics to be a system of knowledge are more consistent with cognitive theory (e.g., Anderson, 1983, 1990; Ausubel, 1968; Gelman & Greeno, 1989; Marshall, 1988; Ohlsson & Rees, 1991), empirical evidence (e.g., Chi & Ceci, 1987; Chi, Hutchinson, & Robin, 1989; Hiebert & Carpenter, 1992; Mayer, Larkin, & Kadane, 1984), and the views of an increasing number of mathematicians and educators (e.g., Davis & Hersh, 1981; NCTM, 1989; Webb & Romberg, 1992).

Theoretical Framework

One concern in this study has been to engage a theoretical framework powerful enough for the analysis of elementary mathematics concepts. Classical concept formation theory does not adequately account for the construction of fundamental math concepts (Ernest, 1991; Glasersfeld, 1989; Herscovics & Bergeron, 1993; Johnson-Laird, 1988; Lakoff, 1988; Wittgenstein, 1968). As Herscovics and Bergeron (1993) observe, it is hard to imagine teaching even the most basic concepts, such as number or function, simply by presenting their attributes for students to memorize, or by listing examples and non-examples. Knowledge about mathematical concepts cannot be reduced to lists of defining attributes, as in classical concept formation theories, and memorization of attributes and exemplars is unlikely to lead to greater understanding than rote acquisition of procedures. At the same time, assessment of piecemeal acquisition of knowledge does not give a valid account of mathematical understanding.

It is now widely acknowledged that mathematical concepts, like other complex concepts, are not optimally characterized as single discrete ideas but as structured systems of knowledge and skill. (Ernest, 1991; Glasersfeld, 1989; Herscovics & Bergeron, 1993; Johnson-Laird, 1988; Lakoff, 1988; Wittgenstein, 1968). Concepts exist in and are characterized by systems of relations to other concepts, each concept serving as an axis around which other forms of knowledge can be organized. The fundamental problem in mathematics education is therefore mastery of a system of concepts, and the essential problem of assessment is to ascertain the degree and quality of that mastery. Thus the NCTM *Standards* and other reform documents stress that assessment strategies should be aligned with the complex structure of mathematics: "Consideration of the structure of mathematics in constructing assessment methods affects how tasks are designed and chosen, how tasks are administered, the desired form of response, [and] what rules are followed to make judgments about responses" (Webb & Romberg, 1992, p. 45). Unfortunately, this alignment rarely occurs. At present there are few published reports of assessment or instructional strategies based on systematic models of mathematics knowledge, despite the fact that most contemporary accounts of mathematical understanding incorporate such models. And in many areas of mathematics, the essential theoretical and empirical work needed to determine what it means to understand important concepts and principles and how that understanding may be developed and assessed in classrooms has yet to be done. As a result, traditional instruction and assessment have not much attended to the structured nature of mathematical understanding, and educational practice is not linked as yet to this powerful theoretical framework. Many widely used large-scale tests, for instance, do not adequately assess understanding of specific concepts or relational knowledge of any kind and are not useful for decision making in the course of teaching for understanding. Their validity rests primarily on "face validity" judgments of content, on correlation with similar tests, on "predictive" correlations, and on post-hoc analyses of the factors represented by different items, not on diagnostic utility or construct validity of the measures. Traditional content and predictive validation procedures provide only limited evidence about the construct-interpretation and use of scores (Messick, 1989), however; many standardized achievement tests have been "validated" by these methods, yet fall short as useful measures of understanding.

When mathematical understanding is conceived as a personally-constructed system of relations among symbols, concepts, operations, and objects or situations, valid assessment of understanding for instructional purposes requires that one obtain information on students' knowledge about each of these entities, as well the relations among them. For example, one would expect of students who had constructed mathematical meaning for the representations and situations they had encountered in school that they would be able to: (a) see that there can be multiple ways to represent the same concept or mathematical structure; (b) see that the structure of a task may be the same despite changes in representation—different representations do not necessarily imply a different concept or operation; (c) use representations of concepts effectively in problem solving; and (d) be able to explain the meaning and use of concepts and representations.

Identification of a Domain for Study

Fractions were chosen as the domain in which to address the objectives of this study for reasons that include the following: (a) fractions constitute one of the most complex and important domains in elementary school mathematics; (b) fractions are among the domains targeted as conceptually important but underrepresented in contemporary curricula, according to the NCTM *Standards* and state curriculum frameworks (e.g., California Board of Education, 1991); (c) few children at any grade level appear to know what fractions are or what fraction symbols represent, a consistent finding in both cognitive research and assessment contexts (e.g., Carpenter et al., 1988; Dossey et al., 1988; Hart, 1981; Hope & Owens, 1987; Kerslake, 1986; Kieren, Nelson, & Smith, 1985; Lesh et al., 1983; Nik Pa, 1989; Peck & Jencks, 1981; Post, 1981; Silver & Carpenter, 1990); (d) the inability to develop fraction understanding inhibits many students from mastering mathematics beyond the elementary school level (e.g., Hart, 1981; Hope & Owens, 1987; Kerslake, 1986; Kieren et al., 1985; Kouba, Carpenter, & Swafford, 1990; Nik Pa, 1989; Behr, Harel, Post, & Lesh, 1992); and (e) evolving theories of mathematical understanding, recent cognitive research, and analyses of the fraction concept suggest new possibilities for addressing these problems.

Although its aim was to contribute to the improvement of mathematics assessment and instruction in general, the study focused on fifth-grade students' fraction understanding. Fifth grade is a critical year because for many students (in California) it is the last year of elementary school. And fifth grade represents a turning point in fraction instruction. In kindergarten through fourth grade, children typically encounter a bewildering array of fraction representations and meanings, including parts of pies or brownies; subsets of sets of objects; "sharing" situations; parts of line segments; fractions as operators, quotients, ratios, proportions, probabilities, and measures; and decimal fractions. The implicit expectation is that students will induce fraction understanding as a result of exposure to many different representations, models, and activities. The crucial question is whether students *are* able to generalize across these disparate experiences to construct a coherent understanding of fractions and fraction representations, one that serves as a basis for understanding operations on fractions, and for making sense of new situations and tasks. This is a question of singular consequence, because a review of commonly-used instructional sequences (e.g., Eicholz, O'Daffer, & Fleenor, 1989; Fennel, Reys, Reys, & Webb, 1988; Scott, Foresman, 1988) shows that, after fifth grade, fraction instruction becomes increasingly and more explicitly procedural, emphasizing computational algorithms rather than the meaning of the fraction concept. The NCTM *Standards* (1989) recommend such a sequence: kindergarten-through-fourth grade fraction curricula should focus on meaning, and fifth-grade and middle school curricula should cover operations and the relation of fractions to other constructs such as decimals. On the evidence, most schools adhere to this sequence, so if students do not know the meaning of fractions by fifth grade, they are unlikely to get many more opportunities to develop that understanding.

Objectives of the Study

The purposes of this study were to develop and test related assessment and instruction activities based on analysis of a pivotal concept in elementary mathematics (fractions), to ascertain whether the assessments were responsive to variations in instructional content, to examine whether the assessments provided diagnostically useful information, and to explore relationships among measures of different types of knowledge and performance, including external validity measures.

REVIEW OF LITERATURE

Children's Difficulties With Fractions

Fraction knowledge forms a basis for understanding a wide range of related concepts, including ratio, proportion, decimals, per cents, and rational numbers; and it is essential to expertise in more advanced topics such as algebra and calculus (Kieren, 1992). Because of their importance and difficulty, fractions are conventionally introduced to children in kindergarten and occupy a prominent place in school curricula from second grade on. In typical textbook series (e.g., Eicholz et al., 1989; Fennel et al., 1988; Scott, Foresman, 1988), 25% or more of the intended instructional material in Grades 2 through 6 covers fraction symbols and related rational number representations.

Yet even the simplest fraction problems vex many students, as analyses of large-scale assessment data show. Fifty-five percent of the 13-year-olds in one national sample selected either 19 or 21 as the best estimated answer to 12/13 + 7/8, and 30% added numerators and denominators to find the sum of 1/2 and 1/3 (Post, 1981). In the fourth National Assessment of Educational Progress (NAEP), only 44% of eleventh graders could choose the correct answer for the following item (Carpenter et al., 1988, p. 40):

5 1/4 is the same as:

a. 5 + 1/4

b. 5 - 1/4

c. 5 x 1/4

d. 5 / 1/4.

NAEP results indicate that many students see fractions as purely symbolic entities not linked to concepts or principles. Several findings support this conclusion. Performance is much higher on problems where known algorithms are easy to apply than on problems that are relatively difficult to solve using common algorithms (e.g., 3 1/2 - 3 1/3 versus 7 1/6 - 3 1/2). But there is no corresponding effect related to the ease with which conceptual knowledge might be used to solve a problem, suggesting that many students do not have or do not use conceptual knowledge to answer NAEP items.

In more complex problem-solving situations, where greater understanding is required, performance is even more egregious. Studies across a range of grade levels confirm that many students resort to searching among procedures that have almost no meaning to them in order to find one that might lead to an answer, and cannot evaluate or justify their solution procedures (Behr et al., 1992; Behr, Lesh, Post, & Silver, 1983; Behr, Wachsmuth, & Post, 1985; Hart, 1981; Kerslake, 1986; Kieren, 1988; Nik Pa, 1989; Pandey, 1991; Peck & Jencks, 1981; Post, 1981; Silver, 1981; Wearne & Hiebert, 1988a, 1988b). In one study, 90% of several hundred sixth-through-eighth-grade students interviewed could not give any meaningful explanation or representation for fraction addition and were judged to have a conceptual base inadequate to guide problem solving or further study of fractions (Peck & Jencks, 1981). When asked to sketch fractions, nearly all students could show 1/2 but fewer than half could represent fractions such as 3/5

(Peck & Jencks, 1981). Researchers in another extensive and widely-cited project found that fourth-through-eighth-grade students' fraction knowledge was extremely unstable and susceptible to almost any type of perceptual distraction (e.g., Behr et al., 1983, 1992; Lesh et al., 1983).

Taken together, these studies document profound deficiencies in children's fraction understanding, and the glaring failure of instruction to address those deficiencies. In consequence of these deficiencies, the NCTM (1989) and other influential professional and government groups have called for mathematics education to move toward a greater focus on mathematics understanding. Yet the statements published by these groups contain a relatively small number of examples of recommended classroom activities and assessment tasks. None of them provide detailed conceptual analyses of any domain in mathematics, nor specific information on how to help students construct conceptual understanding, nor specifications for assessments that might help to ascertain whether students have constructed such understanding. The strategy for addressing these issues in this dissertation begins with an analysis of what it means to understand the fraction symbol.

As the mathematician Rene Thom (1973) has argued, "The real problem which confronts mathematics teaching is not that of rigor, but the problem of the development of 'meaning,' of the 'existence' of mathematical objects" (p. 202). The concept of "meaning" is far from transparent when applied to the term "fraction," however, which no doubt accounts for much of the difficulty of teaching and learning about fractions. Despite the proliferation of efforts to explicate the fraction concept, no single definition of fractions has been identified and used consistently in the literature, (e.g., Behr et al., 1983, 1992; Clements & Campo, 1990; Hope & Owens, 1987; Hunting, 1984; Kieren, 1980, 1988; Larson, 1979; Nik Pa, 1989; Novillis, 1976; Ohlsson, 1988; Pothier & Sawada, 1983). In some cases investigators use the same label for different constructs; in others, different labels for equivalent constructs. Sometimes fractions are equated with rational numbers (Clements & Campo, 1989), while in other reports they are called "subconstructs" (Behr et al., 1983; Kieren, 1980), "applications" (Ohlsson, 1988), or "elements of equivalence classes" (Behr et al., 1992) of rational numbers.

Recent analyses of the semantics of rational numbers reveal that fractions are part of a family of "subconstructs" of rational numbers that includes parts-of-wholes, decimals, ratios, rates, quotients, operators, and measures (Behr et al., 1983; Kieren, 1980). When the symbol a/b represents a fraction, it can be added to other fractions. When it represents a ratio, it cannot be added in the same way to other ratios. This confusing "multi-representationality" complicates the process of learning the meaning of the fraction symbol, at least to the extent that students are exposed to different subconstructs. At least one researcher has resigned himself to the unfortunate ambiguities of fraction usage, arguing that it would be futile to try to standardize the vocabulary related to fractions and rational numbers (Vergnaud, 1983). In this study it has been assumed that it is all but futile to try to teach for, assess, and validate assessments of mathematical understanding without explicating the meaning of the concepts, principles, and procedures to be understood.

Sense and Reference in Fraction Semantics

In an argument paralleled by Detlefsen's (1986) thesis on ways to validate knowledge claims in mathematics, Ohlsson (1987) asserts two sources of meaning for mathematical constructs. First a construct may acquire meaning from the formal theory in which it is embedded: "The axioms and theorems of the theory function as meaning postulates that specify the mathematical meaning of the construct" (Ohlsson, 1987, p. 61). Second, constructs may acquire meaning from their applications: "Applications confer both sense and reference, sense being specified by a natural language concept that circumscribes the class of real-world situations to which the construct is applied. The reference is specified through a mapping between the math construct and real-world objects" (Ohlsson, 1987, p. 61).

Since many formal theorems are advanced achievements occurring relatively late in the history of mathematics (e.g., the laws of the number system and the definition of numbers as entities obeying those laws), these theorems can hardly be taken as starting points for developing arithmetic understanding. Historically, understanding of arithmetic preceded formulation of the laws of number, and this ordering has been sensibly recommended by mathematics educators and researchers since at least the time of Dewey (McLellan & Dewey, 1895). It is implausible to expect, for example, that students can construct a formal number theory before learning counting principles and the laws of arithmetic (Ohlsson, 1987). The recommended strategy is therefore to begin with activities, objects and situations that enable concept development and provide referential meaning for mathematical symbols (a course recommended by Brownell, 1945; Clements & Campo, 1989; Davydov & Tsvetkovich, 1991; Glasersfeld, 1987, 1989; Goldin, 1987; Hiebert & Carpenter, 1992; Lampert, 1986; McLellan & Dewey, 1895; Menchinskaya, 1969; National Council of Teachers of Mathematics, 1989; Ohlsson, 1987; Resnick, 1987, 1989; and many others). One effect of learning to understand the language of arithmetic in this way is that “procedures and principles become easier to learn and to understand” (Ohlsson, 1987, p. 319). For this reason, the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards* recommend that school curricula should focus on conceptual understanding:

> A conceptual approach enables children to acquire clear and stable concepts by constructing meaning in the context of physical situations and allows mathematical abstractions to emerge from empirical experience. A strong conceptual framework also provides anchoring for skill acquisition. Skills can be acquired in ways that make sense to children and in ways that result in more effective learning. A strong emphasis on mathematical concepts and understandings also supports the development of problem solving. (1989, p. 17)

Opportunities to develop referential meaning have another function, which is to demonstrate that the laws of arithmetic are “true,” “to give intuitively convincing demonstrations, in lieu of the strict proofs which presumably should take their place when the learner has reached a more mature age” (Ohlsson, 1987, p. 311).

Importance of Symbolic Understanding

Compelling arguments for providing a referential semantics have been made on the basis of the inherent “representationality” of mathematical activity. The burden of the argument is twofold: (a) symbols are indispensable to mathematical activity (Davis & Hersh, 1981; Goldin, 1987; Kaput, 1987; Pimm, 1987; Resnick, 1987; Skemp, 1982); and (b) mathematical symbols have meanings and may be interpreted; they are not just marks that can be manipulated to perform calculations (Davis & Hersh, 1981; Skemp, 1982). With respect to the second point, it has been noted that many mathematical symbols have multiple meanings or referents, permitting a great variety of situations to be represented by a small number of symbols. For example, the symbol 2/3 represents a rational number which in turn can represent diverse situations, including those involving ratios, rates, proportions, and division. Understanding the applications of a symbol means that one can abstract the elements that different problem situations have in common and can represent these elements by mathematical symbols, giving access to solution procedures that might otherwise be unavailable.

Historically symbols and representations have played an essential role in the development of mathematical thought. Descartes, for instance, argued that symbolic writing made it easier to keep mathematical elements in mind, permitted the external representation and visualization of ideas, and enabled mathematicians to organize ideas and patterns of reasoning more succinctly and develop broader intuitions (Bednarz, Dufour-Janvier, Poirier, & Bacon, 1993). Mathematical symbols can also be subsumed under the general claim that explicit representation of knowledge "provides an intellectual tool of great power, generality, and cultural significance" (Bruner & Olsen, 1977-78, p. 12). Mathematical symbols hardly constitute "an intellectual tool of great power" in elementary school, however, where most students and possibly some teachers do not understand what the symbols they are using mean.

Fractions as Pieces of Pie

One particular semantic referent or model has dominated elementary fractions instruction. In this model, the denominator of a fraction is interpreted as the total number of parts in some object (often a pie) or set of objects, and the numerator as a number of those parts selected for some purpose. This is a "part-whole" model of fractions (Behr et al., 1983; Kieren, 1980, 1988; Novillis, 1976). Part-whole illustrations make it easy to generate language about fractions because already-acquired whole number language can be used. Part-whole models can also be easily assimilated to counting schemas, particularly when all examples presented to students are "already divided up." Exclusive reliance on part-whole illustrations risks leading many students into misconceptions, however. Fraction numerators and denominators can be seen in these situations as unrelated whole numbers representing two separate counts. Children simply learn to put the outcome of one count (number of pieces in the "part") above a "fraction line" and the outcome of the other count (number of pieces in the "whole") below the line. A fraction such as three-fourths is then regarded only as the outcome of a double count and not as a single number or quantity. This makes it more difficult to conceive fractions as numbers. Under this model, fractions greater than one are conceptually anomalous. (In fact most textbooks refer to fractions such as 7/6 as “improper”; 1 1/6 is not called a fraction but a “mixed number.”)

Both Davydov (Davydov & Tsvetkovich, 1991) and Dewey (McLellan & Dewey, 1895) have criticized part-whole instruction for its reliance on “vague percepts” in place of concepts or principles. Other critiques of the part-whole model spring from the fact that this model does not generalize well to other fraction applications and rational number subconstructs. A fraction is not “a piece of pie that you eat,” but an abstraction, a number that represents a relation between two other numbers: “The thought of 7/8 demands the thought of both numbers, 7 and 8, and the thought of their modification each through the other” (Harris, 1895, p. vii).

“To begin the teaching of fractions with vague and undefined ‘units’ obtained by breaking up equally undefined wholes—the apple, the orange, the piece of paper, the pie—may be justly termed an irrational procedure,” McLellan and Dewey (1895) argue. “Half a pie, e. g., is not a numerical expression at all, unless the pie is defined by weight or volume; the constituent factors of a fraction are not present; the unity of arithmetic is ignored; the process of fractions is assumed to be something different from that of number as measurement; it becomes a question—it actually has been questioned—whether a fraction is really a number” (McLellan & Dewey, 1895, p. 140).

Data support the conclusion that it is difficult to develop the concept of a fraction as a number representing a relation between two numbers (or two quantities) simply by partitioning, “shading in,” or combining parts of objects (e.g., Kerslake, 1986; Kieren, 1992). These activities tend to lead to perceptions of fractions as pieces of objects, that is, away from, not toward, a more advanced rational number concept (Harris, 1895). McLellan and Dewey advance the claim that “the definitions which ignore fractions as a mode of measurement are in general vague and inaccurate, and lead to much perplexity in the treatment of fractions. It is hardly accurate to say that a ‘fraction is a number of the equal parts of a unit,’ or that ‘it originates in the division of a *unit* into equal parts.’ These definitions overlook the important distinction between unit and unity. A pie as typically used is not a unit of measurement nor does it become one after it has been cut into pieces. After measuring its surface area, this area can be regarded as a unity of units, or sum” (1895, p. 132). It has also been argued that the part-whole model does not yield a good theory for fractional quantities because among other things it implies that the result of joining 1/4 and 2/4 is 3/8 (Davydov & Tsvetkovich, 1991; Kerslake, 1986; Ohlsson, 1988). This is an extremely common misconception, held by many children throughout elementary school. It is not unusual for students to offer part-whole diagrams as "proof" that a/b + c/b = (a+c)/(b+b) (Kerslake, 1986), as shown in Figure 1.

Kieren (1992) points out serious discontinuities in the traditional sequence for fraction instruction, which begins with the assumption that fractions are parts of wholes and makes several quantum leaps to teaching procedures (such as adding unlike fractions) that are based on number theory principles. The full sequence has five levels:

Level 1: Fractions are assumed to be parts of wholes.

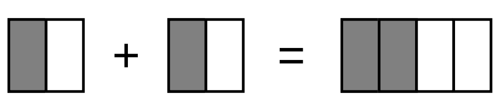
Level 2: Fractions are generated by counting parts in a single predivided whole.

Level 3: Fractions in general are assumed to represent double counts (as opposed, for example, to representing a relation between two quantities); they are generated by counting the number of pieces in a whole and the number of pieces in some part of the whole.

Level 4: Fractions with like denominators are added by counting like "parts."

Level 5: Unlike fractions are added according to number theory principles. This requires finding common denominators, using equivalence principles.

It is presumptuously thought that this sequence represents a hierarchy. In fact, as Kieren (1992) has pointed out, knowledge developed early in this sequence has little bearing on the procedures taught later; for example, the idea of a fraction as a part of a pie cannot be used to justify the procedure for adding 4/3 and 7/5. (Among other reasons, one cannot take 4/3 of a single pie.) To cope with these discontinuities, some teachers give mathematically inapt justifications, such as "You can't add thirds and fourths because that's like adding apples and oranges."



$$\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$$

Figure 1. Using part-whole figures to justify an incorrect addition procedure.

The concept of fractions as fractional quantities implies an alternative to part-whole instruction, specifically one based on measurement applications. It also implies that assessment should not be limited to part-whole models but should encompass understanding of fractional quantities.

Fractions as Quantities

Understanding fractions as quantities implies the construction of several related principles, including but not limited to the following:

1. Principles for measuring quantities, such as:
   (a) The "equal interval principle": all intervals or units in a measured or partitioned quantity must be equal.
   (b) The size of the units used in measurement does not affect the quantity measured but is inversely related to the outcome of the measurement.

2. Any quantity can be measured by some smaller quantity or partitioned into smaller quantities.

3. Between any two numbers you can find an infinite number of fractions.

4. For any given fraction you can find an infinite number of equivalent fractions.

5. Any two quantities of the same type may be compared by measurement. One quantity may be identified as a referent quantity and the other expressed as a fraction of the first. Davydov and Tsvetkovich (1991) have argued that fraction understanding implies the ability to establish the units necessary to carry out this operation.

6. Two quantities may be easily compared or added if they have been expressed in terms of the same measurement unit.

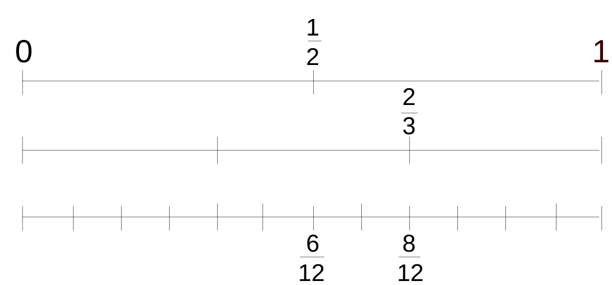
7. Quantities cannot be directly compared, added, or subtracted unless they have the same units of measure.

All of these principles can be developed and understood in the context of measurement activities, and as a set they have been consistently targeted by researchers studying quantitative understanding (e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Bright, Behr, Post, & Wachsmuth, 1988; Davydov & Tsvetkovich, 1991; Gelman, Cohen, & Hartnett, 1989; Kerslake, 1986; Larson, 1979; Muangnapoe, 1975; Novillis, 1976). Among the important ideas that can be derived from these principles about quantities is the concept of fraction equivalence. For example, the relation between two quantities can be expressed in an infinite number of ways simply by changing the unit used to measure the quantities. If one length is found to be 3 inches and another 5 inches, the relation of the first to the second is 3/5. If the same lengths are measured by one-half-inch units, the relation becomes 6/10, and so on. Equivalence is an extremely important feature of fractions, underlying many operations, and has been identified as the "fundamental property" of fractions (Davydov and Tsvetkovich, 1991).

Fraction addition and other operations can also be derived from measurement situations. The procedure for adding 2/3 and 1/2 can be understood in terms of re-measuring the three quantities (the two fractional quantities and the "whole" or referent quantity to which they are related) with a smaller unit, in this case a unit that is 1/12 of the whole quantity. Figure 2 illustrates this re-measurement. After 2/3 and 1/2 have been expressed in terms of twelfths, the two fractions can be added. The challenge in this situation is finding a new unit that can be used to express the two fractional quantities.

Research on Measurement Applications of Fractions

One of the instructional approaches designed for this study focused on the above principles. This approach was modeled in part on successful fraction instruction demonstrated by Davydov and Tsvetkovich (1991). These authors



$$\frac{1}{2} = \frac{6}{12} \text{ and } \frac{2}{3} = \frac{8}{12}$$

$$\text{so } \frac{1}{2} + \frac{2}{3} = \frac{6}{12} + \frac{8}{12}$$

Figure 2. Expressing 1/2 and 2/3 as twelfths.

reported a high level of fraction understanding among third-grade students in response to their method, which emphasized the development of fraction language, principles and symbol use in the context of measurement activities. Their method avoided some of the pitfalls inherent in part-whole models. The search for a common denominator, for example, was related to finding a common measurement unit with which to express quantities. However, results reported are mainly anecdotal; there are no statistical tests, and even when percent of items correct on pre- and posttests is given, the number of students tested is not stated. To assess fraction understanding, Davydov used number lines, measurement, finding equivalent fractions, and finding fractions between two numbers tasks.

In another study, Sambo, cited in Kieren (1992), demonstrated the effectiveness for Nigerian seventh graders of a measure-oriented approach to fraction equivalence and addition. Sambo's strategy derived from the isomorphism between the mathematics of linear measure and rational number but Kieren's description of the interventions used in Sambo's unpublished dissertation was not detailed enough to be useful here. Nevertheless, development of the assessment and instruction for this study was stimulated by the results of these studies, as well as those from related research on number lines.

Research on Quantitative Understanding

Number Lines

Several researchers in addition to Davydov and Tsvetkovich have taken the ability to recognize or place fractions on number lines as evidence of understanding of the quantitative nature of fractions. Larson (Larson, 1979; Novillis, 1976) found in a series of studies that elementary and middle school children have difficulty identifying or placing fractions on number lines, particularly when the number lines show more than one unit. Associating a proper fraction with a point on a number line proved more difficult for intermediate grade students than associating a proper fraction with a part-whole model where the unit was a geometric region (Novillis, 1976) and with a part-group model where the unit was a set (Novillis, 1976). Larson (1979) contends that the difficulty of number lines is attributable to the difficulty of identifying the unit when multiple units are represented; the unit cannot be taken for granted as it can in partitioning a geometric region or set. This point may also account for her finding that for seventh graders associating proper fractions with points on number lines of length one was significantly easier than associating proper fractions with points on number lines of length two. Another result Larson obtained is that associating proper fractions with points on number lines whose units were subdivided into segments equal in number to the denominator of the fraction to be represented was significantly easier than with points on number lines where the number of segments did not equal the denominator.

These tasks should have been equally difficult regardless of the number of partitions or whether the number lines represented one or two units. Difficulties suggest that students were mechanically applying part-whole concepts or other inappropriate routines and were lacking knowledge of number line representations, a concept of proper fractions as naming a number of equivalent parts of a defined unit, or a concept of fractions as names for numbers (Larson, 1979). A related point was made by Muangnapoe (1975) who found that third and fourth graders tend to treat a total number line as a unit no matter how many units are actually represented. Streefland (1978) saw the number line as a critical device for helping students to develop a coherent fraction model, but argued that it should be used as a tool for formalizing experience and organizing results from action-oriented experiences. This approach is closer in spirit to the instruction developed for this study than that of other researchers who have studied whether students could learn specific procedures for placing fractions on number lines. Bright et al. (1988), for example, found that 4–8 days of such "procedural" instruction can be effective in improving performance on number line tasks.

Effects of Fraction Familiarity and Size

Salim, in a study reported in Vergnaud (1983), found that fraction understanding was strongly related to the numerical value of the fractions. The difficulty of three different expressions of the fraction relation ("find the compared quantity," "find the operator," "find the referent quantity") was expected to vary, as was the difficulty for discrete (set of pearls) and continuous (discs and strips) quantities. There were some differences, but Vergnaud (1983) described them as "not as large as expected and very small compared with the differences due to the numerical values" (p. 168). All tasks were easily accomplished for both continuous and discrete applications of 1/2 and 1/3, but the same tasks with other fractions, particularly non-unit fractions (numerator $\neq 1$), were difficult for many students.

For unit fractions (numerator = 1), there were only "slight" differences on the three relational variants; differences for non-unit fractions were greater.

Salim found a four-year gap between mastery of 1/2 and mastery of other unit fractions on most tasks; that is, mastery of tasks involving 1/2 occurred up to four years earlier than mastery of tasks involving other unit fractions. 1/4 was the next fraction mastered after 1/2; 1/3 proved more difficult than other unit fractions. In the first grade, 1/2 was "fully mastered" by 30% of the subjects and "partially mastered" by 50%. By the fifth grade, all students had mastered 1/2 but only 40% had mastered 3/4 and 2/5. On a simpler task, where students were presented disks (partitioned by lines) and asked to show 1/2, 1/4, 1/5, 3/4, and 2/5, 50% of the first grade students succeeded with the last two fractions. Their strategy was simply to count pieces of the disk, e.g., three pieces for 3/4. A similar strategy worked for comparing 3/5 and 2/5; many first graders said that three of something was more than 2 of something. These are whole number strategies that work on certain fraction tasks but not others. First and second graders, for example, could not successfully compare fractions when the denominators were different because a larger denominator means a smaller fraction. When one is interested in whether students have a mathematically correct fraction concept, and are not simply using whole number principles to solve problems, the choice of tasks is critical.

In another sequence of studies investigating children's rational number ideas in Grades 2–8, Lesh (Lesh, 1981; Lesh et al., 1983) found that task characteristics such as number size, context, and type of manipulative material caused performance variations, suggesting that many students have unstable models. For example, word problems differed from analogous real-world problems in difficulty, predominant representational mode used in solution, and most frequent error types. It is difficult to draw definite conclusions from Lesh's studies, however, because task characteristics are varied unsystematically and statistical tests are not reported.

Assessing Fraction Understanding as a System of Knowledge

To capture the embeddedness of mathematical concepts in organized knowledge structures, several different models and theories have been developed in the last 15 years or so. Mathematical knowledge has been variously conceived in terms of relational or schematic knowledge (Skemp, 1972, 1976), conceptual fields (Vergnaud, 1983, 1988; Webb & Romberg, 1992), conceptual schemes (Herscovics & Bergeron, 1993), and theories about concepts (Detlefsen, 1986; Kitcher, 1983; Ohlsson, 1988). These highly related constructs, two of which are considered in more detail below, coincide in highlighting the central status of mathematical concepts in organized knowledge structures.

Conceptual Schemes

Herscovics and Bergeron (1993) have introduced the expression "conceptual scheme" to refer to relational structures or networks in a given concept domain. This phrase refers to the notion of a network of related knowledge about a concept together with the situations where such knowledge can be used. Analysis of a concept scheme provides the basis for designing assessment tasks and instructional activities. The notion of a conceptual scheme is fundamentally consistent with contemporary versions of schema theory (e.g., Anderson, 1990; Chi & Ceci, 1987; Chi et al., 1989; Marshall, 1988), as well as other approaches to understanding that discuss the integration of skills and concepts and the advantages of such an integration (e.g., Ausubel, 1968; Brownell, 1967; Gelman & Greeno, 1989; Hiebert & Carpenter, 1992; Mayer et al., 1984; Ohlsson & Rees, 1991). Schemas have been thought of as organizing and relating both declarative knowledge—"knowledge about facts and things"—and procedural knowledge—"knowledge about how to perform various cognitive activities"—(Anderson, 1985, p. 198) as well as the strategic (Greeno, 1978; Linn, Baker, & Dunbar, 1991; Messick, 1984) or interpretive knowledge (Gelman & Greeno, 1989) that facilitate problem solving; and linguistic/symbolic knowledge that enables explicit expression of concepts and principles (Brown, 1987). Numerous related analyses of expert knowledge focus on knowing the concepts and principles that organize the domain and represent another way of referring to schema constructs (e.g., Gelman & Greeno, 1989; Ohlsson & Rees, 1991; Skemp, 1972).

Across these related theoretical frameworks, building new conceptual understanding can be seen as a process of learning to operate within a conceptual scheme; that is, learning: (a) to operate within the system; (b) to make connections among concepts and between concepts, symbols, and skills; and (c) to use concepts to make sense of situations, solve problems, and generate new knowledge. Point (c) implies that both concepts and the symbols used to represent them can be connected to semantic knowledge. This is consistent with the frequently-made point that mathematics has semantic content. It has often been argued, for example, that mathematical symbols are not just empty tokens that can be arranged and manipulated to perform calculations (Davis & Hersh, 1981; Goldin, 1987; Kaput, 1987; Pimm, 1987; Resnick, 1987; Skemp, 1982). Rather, the marks or symbolic representations can be interpreted as referring to mathematical concepts and operations, which in turn can be applied to practical/concrete/physical/"real-life" objects, situations, and activities.

Several authors have explicitly addressed the general question "How is systematic understanding of mathematics constructed?" from an assessment perspective. Webb and Romberg (1992), for example, outline a strategy for identifying key elements in a structured domain that builds on Vergnaud's (1983, 1988) theory of conceptual fields. A conceptual field has several types of elements: a set of symbols; concepts and operations represented by the symbols; procedures for transforming the symbols; and objects, actions, and situations that give meaning to the symbols and concepts. These elements map directly onto those in the theories of mathematical representation referred to earlier.

Vergnaud's notion of conceptual fields is based on the premise that "a small number of symbols and symbolic statements can be used to represent a vast array of different problem situations" (Webb & Romberg, 1992, p. 45). This means that many different assessment situations can be generated from the small set of symbols that define a single conceptual field. Assessing a conceptual field requires that the defining symbols, concepts, and situations be identified, a process that Webb and Romberg (1992), following Vergnaud (1983), call "constructing the domain." Hively, Patterson, and Page (1968) in effect describe a similar methodology for specifying assessment domains, albeit within a different theoretical framework. Constructing the domain yields a "map" or relational network of domain knowledge. This map provides a framework for assessment task design that is somewhat different from other possible frameworks such as content-by-behaviors. When *students* (as opposed to assessors) construct such a map, it is equivalent to a mental model or schema for the domain.

Analysis of Elements in the Conceptual Domain

Following Vergnaud, a three-step process was used to specify elements in the domain for this study:

1. Identification of symbolic expressions characterizing the domain.

2. Specification of implied tasks in the domain, including symbolic manipulation.

3. Identification of situations that give meaning to concepts represented by the defining symbols, or to operations performed on the symbols.

The symbolic expressions that defined the target domain for this study were the fraction symbol a/b, defined as representing a fractional quantity, and related expressions such as a/b = c/d. Tasks implied by these symbols include, for example, representing fractional quantities and equivalent fractions. Implied situations include a great diversity of contexts in which it is useful to represent and compare or operate on fractional quantities. In the assessment tasks devised for this study, many of these situations were described verbally and pictorially in word or "word-and-picture" problems or were represented graphically in representational knowledge items. In addition, students were expected in their problem-solving justifications and explanations to make connections among symbols, graphic representations, linguistic representations of principles and concepts, and "real-life" situations.

Construct Validation of the Assessments

Construct validity has traditionally been evaluated by "testing what qualities a test measures, that is, by determining the degree to which certain explanatory concepts or constructs account for performance on the test" (Messick, 1989, p. 16). When the goal is to provide broad predictive or evaluative information, as in most large-scale standardized achievement testing, only highly indirect and usually inadequate answers are obtained for the question "Do students understand what they are doing when they do mathematics?" Valid information on understanding of particular concepts is generally not obtainable from measures like these that consist of items sampled from many disparate domains and that are not aligned with schematic conceptions of mathematics understanding. Even Bloom (1956) has argued that testing large numbers of discrete skills and fragments of knowledge "might lead to fragmentation and atomization of educational purposes such that the parts and pieces finally placed into the classification might be very different from the more complete objective with which one started" (pp. 5-6).

Given the considerable evidence on the power of testing to influence curriculum, testing, and learning (Linn et al., 1991; Fredericksen, 1984; Garcia, Rasmussen, Stobbe, & Garcia, 1990; Goldin, 1992; Hatch & Gardner, 1990), it is likely that tests emphasizing fragmented, memorized knowledge encourage teachers and students to see mathematics in this way. It is well documented that students tend to believe that mathematics is the collection of skills they are tested on, and most students believe mathematics consists of a huge number of procedures and facts to be memorized (Izard, 1993). Test use, Messick argues, should be based on the "action implications" of score meanings, and there is an urgent need for validation of assessments aligned with concept-referenced instruction. Diverse types of information can contribute to the validation of such assessments, but the contribution will be strongest where alignment of empirical information with an underlying theory of score interpretation has been explicitly evaluated (Messick, 1989).

In the past, construct validation has emphasized internal and external test structures, or "the appraisal of theoretically expected patterns of relationships among item scores or between test scores and other measures" (Messick, 1989, p. 7). More significant and illuminating evidence may be obtained, however, when a theory of the construct serves as the basis for identifying knowledge and skills to be assessed, for designing the assessment tasks, and for interpreting patterns of performance (Messick, 1989). Where instructional improvement is a goal of assessment, sensitivity to instruction constitutes essential evidence about the interpretation and instructional implications of the assessment results. When the characteristics of an instructional program used to validate assessment tasks are based on a theory of the construct being assessed, both construct theory and tasks gain credibility (Messick, 1989). At the same time, assessments that are insensitive to learning can hardly be used to guide and inform instruction. The key measures for this study were derived from two essential tools for generating meaning—knowledge representation and explanation. Validation strategies for these measures depended on the theoretical analyses of fractions and mathematical understanding reported in other sections of this report.

Hypotheses

Based on the analysis of fractions as a conceptual scheme, the role of explicit representations in the development of understanding, and empirical work cited above, predictions were made about student performance across tasks. Students who received explicit instruction on fraction principles were expected to demonstrate greater understanding of principles across a variety of conceptual, problem-solving, and explanation tasks. It was also expected that these students as a group would be able to identify and generate more representations of fractions as quantities, that is, number line representations. To the extent that students across both instructional groups understood the principles underlying construction of fraction representations, they should also have been able to recognize a greater number of correct representations of fractions and avoid misidentifying incorrect representations, assuming that the range of representation types was sufficiently large. Greater representational fluency was therefore hypothesized to be associated with higher performance on complex tasks requiring understanding and use of fraction representations.

Specific hypotheses tested were as follows:

1. Students who receive explicit instruction on fraction principles (principle group) will demonstrate greater understanding of principles than those who receive activity-oriented instruction (activity group).

2. Students who receive instruction on measuring principles (principle group) will be more fluent in recognizing and constructing quantitative representations of fractions than students who do not receive such instruction (activity group).

3. Students more fluent with fraction representations will have higher performance on explanation tasks.

4. Students with greater fluency in recognizing fraction representations will have higher performance on problem-solving justification tasks.

METHOD

Overview of Research Design and Procedures

To test instructional sensitivity of the tasks, two dissimilar instructional approaches were set up, one designed to teach principles derived from relatively unfamiliar measurement applications of fractions (principle group), the other based on traditional fraction activities found in textbooks (activity group). The textbook activities essentially presented part-whole and operator applications of fractions.

Post-instruction performance of the two groups was compared on measures of representation knowledge and explanation. Data on problem solving, computation, and conceptual/declarative knowledge were obtained as validity evidence—specifically, to check whether there were relationships between these measures and measures of explanation and representation knowledge. Eighteen classrooms, nine in each group, were observed to verify that instruction occurred as intended, and external validity data were collected from teachers and the school district.

Procedures

Twenty-two fifth-grade classrooms were recruited from a midsized urban/suburban school district in a manufacturing city in southern Washington. Altogether 540 students participated although not all students completed all measures. Overall duration of the study was about three weeks.

Before instruction, students took a 20-minute, 60-item pretest. Then all students were given seven and one-half days of instruction. (Overall duration of instruction ranged from 1 1/2 to 2 1/2 weeks because of assemblies, days off for conferencing, and so on.) Eleven classrooms were randomly assigned to a "principled" instruction condition, and eleven to a group receiving instruction based on the district's fifth-grade mathematics textbook. For schools with two or more classrooms, classrooms were randomly assigned within schools, in an effort to balance instructional treatments within schools. Half of the remaining "unmatched" classrooms were then assigned at random to each instructional group. Immediately after instruction both groups spent two class periods completing the posttest measures described above.

Training for Instruction

Teachers in both instruction groups attended training workshops before introducing instruction. Teachers in the principled instruction group met twice for a total of about three hours, while the textbook (activity) group met once for two-and-one-half hours. One teacher missed the activity training and one teacher missed the first of the two principle group sessions; these teachers were briefed by telephone and were not judged to be significantly handicapped by their absences.

(This was especially true for the activity teacher, who was a district trainer on the activity program.)

During the workshops, teachers were introduced to the aims and overall design of the study. A researcher explained to both groups that the purpose of the study was not to evaluate teachers or students but to investigate the effectiveness of some new assessments, and that fractions had been chosen because they were hard for many students to understand. Teachers then reviewed instruction activities and plans for implementing them. The general instructional approach recommended to both groups was whole-class discussion interspersed with individual and small-group activity, as implemented in the Japanese classrooms described by Stigler and Stevenson (1991). The notion of meaningful instruction as described was also discussed with teachers. Meaningful or understanding-based instruction was described as promoting analysis of symbolic expressions in terms of meaningful referents, reducing the need for every syntactic rule and computational procedure to be memorized. If students understand symbols and rules, new rules and procedures can be figured out from known ones.

The goal of this training was not to create dramatic differences in teaching style across groups, but to ensure to the greatest extent possible that instruction for the principle group would give rise to more opportunities for developing mathematical principles. It was assumed that teachers could not drastically alter their teaching procedures or styles after only three hours of training, and that major differences in teacher quality and method across groups would be controlled by random assignment of teachers to treatments. As it turned out, this assumption was supported by data on teacher knowledge, experience, and confidence in teaching math, and by classroom observations (see Analyses and Results). On average, for example, teachers in both groups asked about the same number of questions during instruction, and organized the same number of small group activities. The big differences were in the topics on which questions and activities focused. Principle group questions and activities were directed at explicit representation of principles; activity group instruction was more focused on learning how to do things, for example, learning procedures for figuring out pizza shares, or for computing a fraction of a number. To clarify these differences, differences in methods for training teachers in the two groups will be discussed in somewhat more detail.

Training for Principled Instruction

At the first meeting, held one month before instruction began, measurement applications of fractions and the possible advantages of fraction principles derived from measurement situations were discussed. Teachers had received explanatory materials and brief descriptions of possible activities about one week before the meeting. Principles related to fractional quantities were reviewed. Principles discussed were identical to those listed above under the heading "What It Means to Understand Fractions as Quantities." All teachers agreed that it would be useful for students to experience a new application of fractions, and that measurement applications and related principles were bypassed in the *Real Math* curriculum. Some teachers endorsed the idea that the instruction we were planning could help students to construct a more mathematically-correct fraction concept, but it is not clear that all teachers felt this way. At the second meeting, ten days before instruction began, teachers worked in small groups to evaluate and refine instructional plans, then all teachers reviewed plans and suggestions generated by the small groups.

The purpose of the training was to familiarize teachers with the goals and content of the instructional activities, not to give them detailed, minute-by-minute "scripts" to follow. They were introduced to the concept of a fraction as a relation between two quantities and to applications of this idea in measurement situations. Nevertheless, some teachers requested sample scripts so they could see how the activities might be conducted. Several such scripts became part of a new set of guidelines (see Appendix A) that were sent to teachers one week before instruction began. The guidelines contained descriptions of instructional activities and suggestions for introducing and managing the activities. Student activity sheets were also included for teachers who might want to use them (see Appendix A).

Teachers were advised to improvise and adapt the suggested activities in any way deemed necessary to help their students understand the targeted concepts. Prior to the training workshop, one teacher, recommended by the district's mathematics resource coordinator, tried out the activities for one week with her class and made suggestions for improving them. During the training workshop she recommended the activities highly to other teachers, gave advice on managing group work, and described a fraction game she had invented while testing the activities. This game was added to the curriculum package for all teachers. The teacher who pilot tested activities was not included in the final study.

Activities in the guidelines for instruction were organized into four modules comprising 10 lessons. Lessons 1–7, which all classes completed, covered measuring principles and procedures, partitioning line segments, and placing fractions on number lines; lessons 8 and 9 dealt with fraction equivalence; and lesson 10, adding fractions (Appendix A).

The purpose of the measurement instruction was to ensure that students knew the measurement principles prerequisite to developing a measurement-based model of fractions. Approximately two days of instruction were devoted to measuring with non-standard units, followed by discussion of the advantages of measuring with equal-sized units and the effect of different-sized units on the outcomes of measurement. These measurement activities were added to the sequence in response to teachers who were worried that their students did not know how to use rulers to measure lengths. It is important to note that these were not conventional measuring lessons, however; they were structured to help students understand fundamental principles. In lesson 1, for example, students are asked to measure objects with variously-sized units and to theorize about the effects of measuring objects with different units (Appendix A).

At the suggestion of the district's mathematics resource coordinator, teaching guidelines for the early lessons were more detailed than those for later lessons; she hoped this would encourage teachers to rely more on their own judgment and initiative to plan activities, once they became more comfortable with the content and aims of the lessons.

Only six of the eleven classes completed lessons 8–10, which constituted brief introductions to the possibilities for using number lines to understand fraction addition and equivalence. This was not unexpected, given that teachers reported sizable initial differences in knowledge and ability between classrooms. Because some students could not measure with rulers at the onset of instruction, it was not predicted that all students would be able to justify fraction addition after two days' (at most) experience with number lines.

Activity Training

Activities for this group were selected from the textbook used by all fifth-grade teachers in the district, Open Court's *Real Math* (Willoughby, Bereiter, Hilton, & Rubinstein, 1991). According to the publisher:

> Thinking skills and problem-solving strategies, real applications, mental arithmetic, estimation, approximation, measurement with metric units, organizing data, and topics in geometry, probability, and statistics are emphasized at all levels [K–8]. (Teacher's Guide, Level 5, p. xii)

Computation skills are stressed at particular grade levels and "maintained" or reviewed at later levels. Fractions are introduced at Level 1 (first grade). By the end of Level 4 (fourth grade), the following topics have been covered: introductory work with fractions (shading and identifying parts of figures), fractions of areas and fractions of numbers, fraction-decimal conversions, and addition and subtraction of common fractions. Review of the Level 1–4 texts shows that part-whole and operator illustrations are emphasized: objects are divided up and some parts shaded or taken way to show fractions, and students are further taught that a "fraction is something that operates on other things, including numbers," for example, 2/3 of 24 (Willoughby et al., 1991).

Activity instruction began with several short lessons (approximately 10 minutes each) on fraction notation, estimating fractional lengths, and adding fractions. All classes spent one-and-one-half to two-and-one-half days on each of the following topics:

1. Finding fractions of objects and figures (pizzas, wooden boards, rectangles).
2. Finding a fraction of a number: "1/3 of 15 = n; find n."
3. Finding missing numerators or denominators in equivalent fractions.

Three classes spent all seven-and-one-half days on these topics.

Eight classes (all but three) spent an additional one to two days total on (a) adding fractions with like denominators, after seeing addition modeled on line segments and pizzas; and (b) word problems involving fractions of objects, lengths, volumes and time periods. One class completed all of the above activities and also covered adding and subtracting with unlike denominators, and mixed numbers and improper fractions.

Four lessons scattered throughout the text were assembled into one unit for this study, following the recommendation of a district master teacher who had combined these lessons in previous years and felt they made a cohesive unit. This represented a change from the normal sequence in the text, which is based on the principle that students should spend only a day or two at a time on any given topic. Returning to the same topic later in the year enables students to gradually synthesize understanding over time, according to the district's mathematics coordinator.

An example of an activity group lesson is shown in Appendix B. It is possible that some students developed principled understanding by doing these activities, but it was hypothesized that explicit efforts to develop principles and concepts would produce greater understanding. With respect to familiarity of the content, however, the activity teachers had some distinct advantages over the principled instruction teachers. Most of the activity teachers were teaching lessons they had taught many times before; they averaged about three years teaching from the *Real Math* text, and only one teacher had less than one year of *Real Math* teaching experience. All teachers had attended district workshops on teaching the *Real Math* program. Two of the teachers were school district trainers for these workshops. None of the teachers in the principle group reported prior experience teaching measurement applications of fractions.

Methods for Assessing Principled Understanding

The NCTM *Curriculum and Evaluation and Assessment Standards* (1989, 1993) recommend multiple sources of information about students' understanding, which is consistent with the idea that domains in mathematics have complex internal structures and complex relations with other domains, as Vergnaud (1983) and others cited above have argued. As fraction understanding implies a system of interconnections among symbols, concepts, operations, and meaningful objects and situations, assessment focused on knowledge of the three main categories of relations:

1. Knowing the referents of fraction symbols, which implies knowing the concepts, operations, objects, actions, and situations to which the symbols can refer. One assessment focused on the relations between symbols and graphic representations, each of which in turn can refer to a great variety of "real life" situations. The use of graphic representations made it possible to assess a much broader range of knowledge than use of "real" situations or hands-on activities. Use of graphic representations also enhances the feasibility of assessing representational knowledge at the classroom level or in larger scale studies.

2. Ability to relate and apply symbols and procedures to particular contexts, including problem-solving situations.

3. Ability to justify the use of the symbols and operations, which for most students means the ability to relate symbols and operations to meaningful situations and verbally-expressible knowledge.

4. Ability to explain fractions in terms of mathematical principles and operations that are part of the conceptual scheme for fractions.

Pretest

Prior to the instructional period, a general measure of fraction knowledge was obtained from all students for possible use as a covariate. This test was intended to cover knowledge assumed to be related to the instruction students would receive, including: general conceptual knowledge (e.g., How many thirteenths equal one whole?); fractions of areas; fractions as measures; fractions as points on a number line; and computational knowledge.

Since no existing test was judged to be an adequate measure of these types of knowledge, pretest items were constructed in cooperation with Geoffrey Saxe, Maryl Gearhart, and Elana Joran, (University of California, Los Angeles), who used a similar pretest in another study. To build the test, items were either constructed or adapted from one of two sources: (a) tasks used in research studies of rational number understanding (e.g., Gelman et al., 1989; Kerslake, 1986; Larson, 1979; Lesh et al., 1983); and (b) fourth-, fifth-, and sixth-grade textbooks.

This test comprised 60 constructed-response items. Most items selected had been successfully solved by approximately 40% to 60% of fifth-grade subjects in previous studies, including preliminary testing for this study.

A copy of the pretest is provided in Appendix C.

Posttests

Appendix D contains copies of the posttests described below. Where item orders were counterbalanced, only one form is shown. Administration directions for the two days of testing may be found after the assessments.

Representational fluency. This was a measure of perceived relations among symbolic and graphic representations of equivalent and non-equivalent fractions. Task format was based on a type of item widely used in assessment and research: This item type requires that students select a graphic representation for a given symbol. Similar tasks are commonly used as instructional activities.

To construct this test five different fraction symbols were placed at the top of five different pages. Students were asked to circle all representations showing the same amount as the symbol at the top of the page. Below each symbol, a set of 18 graphic representations was randomly arrayed on the remainder of the page (see Appendix D). A reverse ordering of the array for each fraction was also produced, and each version given to half the students in each classroom. Each type of representation was equally represented across the five pages; for example, for each symbol there were two correct circle and set representations. Figure 3 shows a matrix of all item types. Altogether the five pages contained 90 graphic representations.

The set of targeted fractions, 1/2, 2/4, 2/3, 4/6, and 3/2, varied on several dimensions likely to affect the difficulty of representing the fraction (e.g., Lesh, 1981; Lesh et al., 1983; Nik Pa, 1989; Novillis, 1976; Vergnaud, 1983); familiarity of the fraction (e.g., 1/2, 2/4 compared with 2/3); greater versus smaller size of numerator and denominator in equivalent fractions (1/2, 2/3 compared with 2/4, 4/6); whether the fraction is greater or less than 1 (3/2 compared with other fractions). Three additional considerations influenced the selection of fractions: (a) Numerators and denominators of represented fractions should be as small as possible to reduce counting errors and permit the use of a large number of relatively small graphic representations; (b) the fractions should include both unit (numerator = 1) and non-unit fractions; and (c) equivalent fractions should be represented.

Graphic representations also varied on several dimensions related to the difficulty of identifying the fraction (Bright et al., 1988; Carpenter & Lewis, 1976; Hope & Owens, 1987; Kieren et al., 1985; Lesh et al., 1983; Nik Pa, 1989; Novillis, 1976; Pettito, 1990). These dimensions included number of partitions and type of representation (area, set, length).

“Distractors” in the form of incorrect representations of each fraction were also included; these were generated by partitioning area and linear representations incorrectly, by marking non-equivalent fractional quantities, or by inverting the numerals in the fraction symbol. Distractors were based on common misconceptions about fractions and fraction representations (Behr et al., 1992). For example, a significant number of students do not realize that the partitions in a fraction representation must be equal in size.

	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{4}{6}$	$\frac{3}{2}$
Area of circle					
Area of circle, equivalent					
Area of rectangle					
Area of rectangle, equivalent					
Alternately shaded area					
Set					
Set, equivalent					
Number line					
Number line, equivalent					
Line segment					
Line segment, equivalent					
Area distractor 1					
Area distractor 2					
Number line distractor 1					
Number line distractor 2					
Line segment distractor 1					
Line segment distractor 2					
Line segment distractor 3					

Figure 3. Types of representations used in representational knowledge measure.

Two differently-ordered sheets were generated for each fraction. First, items were randomly assigned to positions on a page, then a reverse ordering was created. For each fraction, each of these orderings was randomly assigned to a different form (A or B) and the forms were counterbalanced within each classroom.

Students were given 20 minutes to complete this task.

Computation. On this measure there were 11 computation items (covering fraction addition, subtraction, and equivalence) used in earlier large-scale evaluations; these items constitute a scale designed to be maximally sensitive to group-level instructional differences (Miller, 1981). Eight minutes were allotted for these items.

Declarative/conceptual knowledge. This measure consisted of 26 short-answer items requiring: measurement knowledge (5 items), ability to place fractions on number lines (8 items), taking a fraction of a whole number (6 items), "sharing" tasks (4 items), conceptual information such as how many ninths equal one whole (4 items), ability to draw multiple equivalent graphic representations (1 item), and ability to use fractions to represent sharing (2 items) and length comparison (2 items) situations. Presentation order of area and linear representations was counterbalanced. Students had 12 minutes to complete the first 23 items; the remaining declarative/conceptual items appeared in the first three pages of the problem-solving and justification measure (items 1–5, 7, 8, 10, and 11).

Problem solving and justification. Students solved six symbolically presented fraction problems requiring them to compare fractions of a distance (3/5 vs. 1/2 of a mile) and of a pizza (2/5 vs. 2/4 of a pizza), evaluate the truth of an addition statement (1/2 + 1/6 = 4/6) and an equivalence statement (9/12 = 6/8), and find fractions between two other fractions. These six problems are subsequently referred to as MILE, PIZZA, ADD, EQUIVALENT, BETWEEN1 AND BETWEEN2, respectively. In BETWEEN1 students had to find a fraction between 1/2 and 3/4; in BETWEEN2 they had to find a fraction between 2 1/2 and 2 3/4.

In each case students were directed to use writing and drawing to show and explain why their solutions were correct. The fraction comparison problems were given first in counterbalanced order, because they represented situations related to the types of instruction students received. The other four problems followed, ordered in the same way for all students. Total time required for these problems and the nine declarative/conceptual items packaged with them was 25 minutes.

Explanation task. Following the problem-solving posttest, students completed a contextualized explanation task. The writing prompt asked students to imagine that they had been recruited to explain fractions on a television show. They were given several questions to help structure their explanations. The format of this prompt was informed by earlier efforts to elicit mathematics explanations from third- and fourth-grade students (see below). This format represents an effort to elicit lengthier explanations and improve variance on several scored dimensions.

There are a number of reasons for assessing explanations in this domain. First, the ability to communicate mathematical knowledge is a major curriculum goal in the NCTM *Standards* (1989). “A person’s knowing of a conceptual domain is a set of abilities to understand, reason, and participate in discourse. . . . Critical components of these sets of practices include the appreciation and use of explanatory ideals that are shared within the community and provide basic modes and goals of explanatory discourse” (Greeno, 1991, p. 176). Second, explanations may provide evidence about explicit understanding of procedures and concepts (Brown, 1987; Kieren, 1990; Kluwe, 1990) and help to determine whether procedures and declarative knowledge have simply been memorized without understanding. Third, explanations may reflect the degree of complexity and organization of domain knowledge.

Pilot testing of assessments. Drafts of the criterion measures and pretest items were pilot-tested to discern whether: (a) item content and format for tasks assessing representational fluency, problem solving, and explanation were appropriate for elementary school students; (b) time allocations were sufficient; (c) responses to problem-solving and explanation tasks could be scored using a version of the CRESST content knowledge assessment rubric (Baker, Freeman, Clayton, 1991); and (d) patterns of relations among task responses would be as predicted by relational conceptions of mathematical knowledge and analyses of the fraction concept (e.g., Bright et al., 1988; Davydov & Tsvetkovich, 1991; Hunting, 1984; Kerslake, 1986; Kieren et al., 1985; Lesh et al., 1983; Nik Pa, 1989; Novillis, 1976; Pettito, 1990; Piaget, Inhelder, & Szeminska, 1960; Pothier & Sawada, 1983).

For these purposes, students at the end of fourth grade were considered representative of beginning fifth-grade students. Thirty to sixty students at the end of their fourth-grade year completed early versions of the measures. Refined versions of the tasks were then given to 20 fifth-grade students who had received brief instruction on fractions as measures. Findings from these studies informed the redesign of several items and construction of new items for the present study.

Scoring

Explanations

To score explanations, raters used a rubric adapted from Baker et al. (1991) and previously pilot-tested for fifth-grade mathematics explanations by Niemi (1993). The rubric had five dimensions: (a) general impression of content quality, GICQ; (b) use of principles and concepts, C/P; (c) knowledge of facts and procedures, FACT; (d) misconceptions, MIS; and (e) integration of knowledge, INT.

Three CRESST researchers scored the papers; one researcher had previous experience with CRESST rubrics and trained the other two. Rater training followed procedures described in Baker, Aschbacher, Niemi, and Sato (1992). Raters were introduced to the assessment tasks and scoring rubric and were given model responses illustrating score points on each of the dimensions to be scored. Appendix F contains a draft of the rubric as well as anchor papers used by raters to score the GICQ and INT scales, and examples of principles, facts and misconceptions excerpted from student papers. Before scoring, raters practiced scoring several sample responses and were tested on five prescored papers to determine whether they were using the rubric correctly. The criterion for proceeding, based on what had been achieved in previous CRESST studies, was at least 65% exact agreement on all scales.

After each rater had scored 20-30 papers, agreement on these papers was checked and additional training provided; that is, serious score disagreements were discussed. Similar retraining occurred after raters had scored about 80 papers.

Raters scored randomized sets of papers. Some sets were double scored. Intrarater scores were obtained by randomly selecting papers from the first 50 or so papers scored by each rater and inserting these selected papers into sets read later. At the time of rating, raters did not know which instruction groups the authors of the papers belonged to.

A total of 197 out of 506 papers (nearly 40%) was scored by at least two raters. Inter- and intrarater reliability and agreement figures for explanations, as well as for other scored responses described below, are reported in the Analysis and Results section.

Problem solving. Solutions to the six symbolically-presented fraction problems (comparing fractions, judging equivalence and addition statements, finding fractions between numbers) were scored as correct or incorrect (score = 1 or 0). Any mathematically correct solution was awarded one point; expressions did not have to be simplified or reduced; for example, 5/4, 10/8, and 1+1/4 could all be scored as correct. Additional score points reflected the presence or absence of graphic or verbal justifications for each of the six problems; students were given 1 point for either a graphic or verbal justification and 2 points for both, creating a possible score of 3 for each problem.

Problem-solving justifications. Two raters scored 401 and 192 of the problem justifications, respectively. Approximately 15% of student responses were selected at random to be scored by both raters.

Representations generated. Types of representations generated across the six problems were also tallied, in seven categories: circle, rectangle, line segment, number line, triangle, set, polygon. Students received one point for each representation used to justify problem solutions. In addition to a point total for each type of representation, a score was computed to reflect the total number of different types of representations (out of seven possible) that each student produced.

Selected response and short constructed response items. Pretest, representation, computation, and conceptual/declarative knowledge items were scored as correct or incorrect (1 or 0). Subscale scores as well as a total score were computed for the representation, computation, and conceptual/ declarative knowledge items. For any item which students answered by generating a number, all mathematically-equivalent expressions of the correct answer were scored as correct; for example, equivalent fractions.

Additional Data

Classroom observations. An effort was made to verify that instructional treatments occurred as intended in each classroom. Eighteen of the 22 classrooms

(9 in each group) were observed at least once for a full class period. Two retired mathematics teachers were trained to do the observing, in addition to the researcher. Observations of classroom processes and resources in use were taken at 5-minute intervals over one or two class periods, using an observation form adapted from one developed in the Apple Classrooms of Tomorrow℠ (ACOT) project. The researcher had previous experience with this form and trained the other observers to use it. Appendix E contains a copy of the form. Artifacts produced by students and teachers during the study were also collected, and teachers were asked to log activities completed each day.

External measures. Some external validity data, including standardized achievement test scores and gender data were obtained. Elementary school grade point averages are not kept by the school district, but teachers rated, on a 1–5-point scale, students' fraction knowledge and general mathematics ability.

ANALYSIS AND RESULTS

This sections opens with a presentation of pretest results, because total pretest score was used as a covariate in many subsequent analyses. Then analyses and results bearing on the hypotheses are discussed, followed by other analyses pertaining to the construct validity and diagnostic utility of the instruments.

Pretest Results

Table 1 shows means for each pretest scale by instruction group. A hierarchical MANOVA with classes nested within instruction groups was

Table 1

Pretest Means and Standard Deviations by Type of Instruction Group

Variables	Principle		Activity		Total items
	Mean	*SD*	Mean	*SD*	
PRETEST TOTAL	27.4382	10.5922	27.3504	9.1453	60
AREA ITEMS	18.9775	6.2492	19.1732	5.2744	26
LINEAR ITEMS	2.4345	1.5651	2.4409	1.5408	17
COMPUTATION	3.3109	4.1960	2.8386	3.4995	7

Note. *n* for Principle = 267; *n* for Activity = 254.

conducted on three pretest subscales—area representations, linear representations, and computation. The first two subscales reflected representation types emphasized in activity and principle instruction, respectively. This analysis showed no difference overall between groups, $F(3, 18) = .54024$, $p = .661$, and no differences on any of the subscales. These results indicate that random assignment produced groups equivalent on prior fraction knowledge as measured by this instrument.

To control for individual differences in prior knowledge about fractions, total pretest score was used as a covariate in subsequent analyses except where noted. Pretest subscales were not used because inspection of bivariate plots revealed that the subscales were not linearly related to many of the dependent variables.

Summary of Posttest Results

Tests of Hypothesis 1: Effects of Principled Instruction on Explanations

One of the most important findings is that students receiving explicit instruction on fraction principles expressed significantly more principles in their essays than activity group students ($M = .7199$ compared with .4135), and fewer misconceptions ($M = 1.9180$ compared with 1.7057), as shown in Table 2. The finding that principle group students expressed a higher level of principled understanding is important because it shows that instruction can enhance understanding of principles in a relatively short period of time, and that written explanations are sensitive to these cognitive changes.

Table 2
Mean Essay Scores by Instruction Group

Essay dimensions	Principle		Activity	
	Mean	*SD*	Mean	*SD*
GICQ	2.0922	.9480	2.0578	.7937
C/P	.7199	.8877	.4135	.6822
FACT	1.2896	1.5168	1.7280	1.5218
MIS	1.9180	.9447	1.7057	.8940
INT	1.6345	.9784	1.6938	.8556

Note. GICQ = General Content Quality, PK = Prior Knowledge, C/P = Concepts/Principles, TX = Text, MIS = Misconceptions, A = Argumentation. *n* for Principle = 243, *n* for Activity = 235.

Since distributions of essay scores were positively skewed for both groups, nonparametric tests of significance were used; results are shown in Table 3.

As a group, principle group students clearly benefited from the measurement- and number-line related principles they learned. Forty-four percent of them expressed one or more principles in their explanations, compared with only 32% of the students in the activity group. In the principle group, 25.9% of students expressed two or more principles, compared with 9.6% in the activity group. Across both groups, the most commonly expressed principles were (a) the infinite density of fractions between two numbers; (b) equivalence principles, especially the principle that for any fraction there are an infinite number of equivalent fractions; and (c) the equal interval principle with respect to measurement. Percentages of students expressing each of these principles were 25%, 26%, and 2%, respectively

Reviewing familiar area representations during instruction appeared to help activity students to use these representations in their explanations, earning them higher scores on the FACT scale than the principle group (M = 1.7280 compared with 1.2896). Score points on this scale were awarded for correct use of representations, as well as procedures and factual statements. Given the body of data showing the difficulty of number lines for students at all age levels (Bright et al., 1988; Larson, 1979; Nik Pa, 1989), it is not surprising that learning about a difficult new representation, number lines, did not appear to help the principle group students as much as practice with area representations helped the activity

Table 3

Results of Mann-Whitney U-Wilcoxon Rank Sum *W* Test on Explanation Scores, by Instruction Group

Essay dimensions	Principle mean rank	Activity mean rank	*u*	*w*	*z*	2-tailed *p*
GICQ	240.56	243.47	28806.5	58189.5	-.2450	.8065
C/P	261.67	221.92	24359.0	53039.0	-3.6092	.0003
FACT	217.14	267.38	23093.0	63903.0	-4.0770	.0000
MIS	255.37	228.35	25895.0	54575.0	-2.3775	.0174
INT	232.28	251.92	26787.0	60209.0	-1.7222	.0850

Note. GICQ = General Content Quality, PK = Prior Knowledge, C/P = Concepts/Principles, TX = Text, MIS = Misconceptions, A = Argumentation. n for Principle = 243, n for Activity = 235.

students on the FACT scale. In fact the ability of the explanation rubric to discriminate in this way between differentially instructed groups lends support to its construct validity.

Reliability of explanation scores. Inter- and intrarater reliability and agreement figures for double-scored papers are shown in Tables 4 and 5. For all rater pairings on all scales, percentage of agreement within one score point was 100%. Alpha coefficients indicate that individual raters and the raters as a group were scoring consistently. As these results were comparable to or better than those in previous CRESST ratings, the remaining explanations were single scored.

Lower percentages on the FACT scale may reflect the open-ended nature of this scale. Raters counted numbers of mathematical "facts" and procedures ranging from 0 to 9. Scores on other scales ranged from 0 to 5 or less.

Table 4
Interrater Agreement on Essay Scores by Dimension

Raters	GICQ	C/P	FACT	MIS	INT
		Alpha coefficients			
1– 2[a]	.9062	.9584	.9692	.9132	.8767
2 –3[b]	.8884	.7641	.9016	.8317	.9629
1 –3[c]	.7725	.7999	.9214	.8511	.6748
		Percentage exact agreement			
1 –2	84.1	89.1	81.2	85.5	84.1
2 –3	72.4	75.9	58.6	75.9	86.2
1 –3	64.4	78.2	64.8	69.0	62.1

Note. GICQ = General Content Quality, C/P = Concepts/Principles, FACT = Facts and Procedures, MIS = Misconceptions, INT = Integration.

[a]n = 138. [b]n = 29. [c]n = 87.

Table 5
Intrarater Agreement on Essay Scores

Rater	GICQ	C/P	FACT	MIS	INT
			Alpha coefficients		
1[a]	0.8444	0.9056	0.9716	0.9111	0.9000
2[b]	0.9467	1.0000	1.0000	0.0946	0.9302
1[c]	1.0000	0.8989	1.0000	0.8840	1.0000
			Percentage exact agreement		
1	77.8	77.8	72.2	72.2	88.9
2	90.0	100.0	90.0	80.0	90.0
1	100.0	90.0	100.0	80.0	100.0

Note. GICQ = General Content Quality, C/P = Concepts/Principles, FACT = Facts and Procedures, MIS = Misconceptions, INT = Integration.

[a]n = 18. [b]n = 10. [c]n = 10.

Tests of Hypothesis 2: Effects of Instruction on Representational Knowledge

The design for testing effects of instruction on representational fluency was hierarchical, with classrooms nested within instructional treatments (Kirk, 1982). Neither group received instruction on the types of items used in this measure, so these items in effect constituted a measure of transfer of knowledge obtained in activity- and discussion-based settings to symbolic and graphic recognition tasks. The principle group, for example, measured lengths and generated linear representations, while activity students worked on “sharing” or “dividing up” problems involving pies and other objects.

It was expected that each treatment group would do better on the types of representations on which it received instruction, and that these differences would “wash out” when subscales were combined to produce a total score. Three subscale scores were therefore used as dependent measures: number line, segment, and area/set. Principle group students should have correctly identified more number line representations, and activity students should have done better on area and set representations. Both groups received instruction on line segments, so no difference was anticipated on line segment items.

Representational knowledge means and standard deviations for the two instruction groups appear in Table 6. Overall there was a significant difference between the groups. Univariate statistics unexpectedly revealed that the principle group had superior performance on number line items and performed just as well as the activity group on area/set and segment items. Table 7 summarizes these results. The relatively large number of items for area/set reflects the great variety of part-whole representations introduced in elementary school. Number line and line segment representations are much more limited, each having essentially one type of representation.

Table 6

Representation Subscale Means and Standard Deviations, by Instruction Group

Variable	Principle[a]		Activity[b]		Total Items
	Mean	*SD*	Mean	*SD*	
NUMBER LINE	9.7396	3.3761	8.2760	2.7183	20
SEGMENT	18.5283	3.1826	18.5160	3.0698	25
AREA/SET	28.4432	7.4699	29.3560	7.1128	45

[a]n = 265. [b]n = 250.

Table 7

Multivariate Tests of Significance for Group Effect on Representation Subscales

Test name	Value	Exact *F*	Hypoth. *DF*	Error *DF*	Sig. of *F*
Pillais	.36470	3.25306	3.00	17.00	.048

Note. F statistics are exact.

Variable	Univariate statistics					
	Hypoth. SS	Error SS	Hypoth. MS	Error MS	*F*	Sig. of *F*
NUM. LINE	240.15324	506.42839	240.15324	26.65413	9.00998	0.007
SEGMENT	0.03580	292.97075	0.03580	15.41951	0.00232	0.962
AREA/SET	114.91743	3535.05443	114.91743	186.05550	0.61765	0.442

Note. *DF* = 1, 19.

It is not possible from these data alone to tell whether both groups or neither group improved on the part-whole items as an effect of instruction, but a comparison of some posttest items with equivalent pretest items indicated that both groups added to their knowledge about part-whole representations (see “Conceptual/declarative knowledge results” below). For the principle group, instruction on measuring applications appears to have influenced their thinking about part-whole representations and situations.

Tests for Hypotheses 3 and 4: Influence of Representational Knowledge on Problem Solving and Explanation

To test hypotheses 3 and 4, predicting that high performance on the representational knowledge task would be related to problem-solving and explanation performance, two groups were constructed on the basis of total scores on the representation task. Students scoring in the top 25% constituted a high-fluency group; those in the lowest 25%, a low-fluency group. Because the distributions of problem-solving and explanation scores for these groups were not normal, scores were compared in nonparametric tests. Means and standard deviations, shown in Table 8, and Mann-Whitney U-Wilcoxon results, presented in Table 9, convey the significant superiority of the high-fluency group on total problem-solving score and all essay dimensions.

Table 8

Problem-Solving and Essay Dimension Means by Level of Representational Knowledge

Variable	Low fluency			High fluency		
	Mean	*SD*	*N*	Mean	*SD*	*N*
PROBLEM TOTAL	1.3063	1.4881	111	5.2051	3.8294	117
GICQ	1.7603	.6939	112	2.5763	.9666	118
C/P	.3020	.5688	112	1.1017	.9281	118
FACT	.9225	1.1374	112	2.1553	1.8411	118
MIS	1.5595	.8340	112	2.2669	.9051	118
INT	1.3363	.5663	112	2.1836	1.1619	118

Note. PROBLEM TOTAL = total problem-solving score. GICQ = General Content Quality, C/P = Concepts/Principles, FACT = Facts and Procedures, MIS = Misconceptions, INT = Integration.

Table 9

Results of Mann-Whitney U-Wilcoxon Rank Sum *W* Test on Problems and Essay Scores, by Level of Representational Knowledge

Variable	Low fluency mean rank	High fluency mean rank	*u*	*w*	*z*	2-tailed *p*
PROBLEM TOTAL	75.35	151.65	2147.5	8363.5	-8.8293	.0000
GICQ	87.13	142.43	3430.5	9758.5	-6.6281	.0000
C/P	87.46	142.12	3467.0	9795.0	-6.8767	.0000
FACT	91.45	138.33	3914.0	10242.0	-5.4960	.0000
MIS	91.88	137.92	3963.0	10291.0	-5.7717	.0000
INT	89.70	139.99	3718.5	10046.5	-6.2562	.0000

Note. *n* for both groups = 120. PROBLEM TOTAL = total problem-solving score. GICQ = General Content Quality, PN = Concepts/Principles, FACT = Facts and Procedures, MIS = Misconceptions, INT = Integration.

Additional Evidence of Instructional Effects

Problem-solving differences. It was not specifically predicted that the two types of instruction would lead to different levels of performance on problem solutions, because both groups received instruction that could have influenced performance on these tasks. The primary intention in administering symbolic problem-solving tasks was to provide an additional source of validity evidence for the explanation and knowledge representation tasks. Nevertheless, non-parametric tests were conducted to see whether there were any group differences: Principle students did better on the pizza and addition problems, and there were no significant differences in mean ranks on the other problems (Tables 10 and 11).

Table 10

Mean Problem-Solving Scores by Instruction Group

Variable	Principle		Activity	
	Mean	*SD*	Mean	*SD*
MILE	.7695	.9070	.6851	.8338
PIZZA	.9218	.8565	.7574	.8503
ADD	.3498	.8113	.1830	.5435
EQUIVALENT	.2346	.6731	.1532	.5488
BETWEEN 1	.3333	.6492	.3489	.6111
BETWEEN 2	.2510	.5950	.2766	.5431

Note. *n* for Principle = 244, *n* for Activity = 239.

Table 11
Results of Mann-Whitney U-Wilcoxon Rank Sum *W* Test on Problem-Solving Scores by Instruction Group

Variable	Principle mean rank	Activity mean rank	*u*	*w*	*z*	2-tailed *p*
MILE	244.15	234.69	27422.0	55152.0	-.8198	.4123
PIZZA	252.72	225.83	25341.0	53071.0	-2.2738	.0230
ADD	247.13	231.61	26697.5	54427.5	-1.9643	.0495
EQUIVALENT	244.10	234.74	27433.5	55163.5	-1.3958	.1628
BETWEEN 1	235.46	243.67	27571.5	57263.5	-.8457	.3977
BETWEEN 2	233.81	245.38	27170.0	57665.0	-1.3029	.1926

Note. n for Principle = 244, n for Activity = 239.

Reliability of problem-solution and justification scoring. One rater scored all problem solutions as correct or incorrect and a second rater scored 66 of these solutions; the second rater agreed with the first on 65 out of 66 (98.5%) problem solutions. At least 40 justifications for each problem type were also scored by two raters; the remaining justifications were single scored. Table 12 shows reliability coefficients and percentages of agreement on justification scores for the double-scored papers.

Representations generated in problem-solving and explanation tasks. In the course of solving problems, justifying solutions, and creating explanations, principle group students drew five times as many number lines (M = .4368 compared with .0861) and more rectangle and line segment representations of fractions than activity students (Table 13). Activity students created more representations falling into other part-whole categories, circle, polygon, set, and triangle. To create their problem-solving justifications and explanations, students obviously tended to use the types of representations they had been instructed on.

Table 12
Rater Agreement on Problem Justification Scores[a]

	MILE	PIZZA	ADD	EQUIVALENT	BETWEEN1	BETWEEN2	Overall
% Exact agreement	92.50	97.50	89.00	93.00	96.00	98.00	94.33
n	40	40	83	81	81	81	

[a] For 2 raters. n = number of papers scored by two raters.

Table 13

Number of Representations Generated to Justify Problem Solutions, by Type and Group

Variable	Principle		Activity	
	Mean	*SD*	Mean	*SD*
GENERATED CIRCLE	4.0903	3.6079	4.9549	3.8567
GENERATED NUMBER LINE	.4368	1.1518	.0861	.3089
GENERATED POLYGON	.0686	.3051	.1066	.4501
GENERATED RECTANGLE	2.5018	3.1926	2.0205	2.4435
GENERATED SEGMENT	1.6318	2.4880	.5287	.9358
GENERATED SET	.3285	.9308	.5205	1.2288
GENERATED TRIANGLE	.0433	.2210	.0820	.3166

Note. n for Principle = 244, n for Activity = 239.

A hierarchical MANOVA on the number of different types of representations used (out of seven) showed no overall difference between the instruction groups. Mann-Whitney tests on the seven different types of representations, which did not have normal distributions, revealed significant differences between instruction groups on circles, line segments, sets, and number lines (Table 14). Predictably, activity students ranked higher on circles and sets, two representations they studied, and principle group students ranked higher on line segments and number lines. This sensitivity to instruction added to the credibility of the problem justifications as measures of learned knowledge.

Reliability of generated representation scoring. One rater coded all representations. A second rater coded 10 papers, achieving 100% agreement with the first rater on these papers.

Conceptual/declarative knowledge results. Tasks labeled as conceptual/ declarative provided an additional check on the effectiveness of instruction. Most of the items were similar to items used in instruction or on the pretest.

Separate analyses were conducted on three different sets of scores. Administration directions required 12 minutes for the first 23 items, but several teachers erroneously gave these items at the end of short math periods, allotting only 5 minutes for the test. This meant that many students did not have time to complete all items. Since these items were counterbalanced as described in the

Table 14

Results of Mann-Whitney U-Wilcoxon Rank Sum *W* Test on Generated Representations, by Instruction Group

Variable	Principle mean rank	Activity mean rank	*u*	*w*	*z*	2-tailed *p*
GENERATED CIRCLE	226.25	267.42	25188.5	64714.5	-3.2289	.0012
GENERATED RECTANGLE	254.31	238.43	28297.0	57700.0	-1.2742	.2026
GENERATED SEGMENT	273.82	218.28	23420.5	52823.5	-4.9086	.0000
GENERATED SET	237.73	255.56	28056.5	61846.5	-1.9895	.0466
GENERATED TRIANGLE	242.81	250.31	29328.5	60574.5	-1.4815	.1385
GENERATED POLYGON	244.70	248.36	29799.5	60103.5	-0.6787	.4973
GENERATED NUMBER LINE	262.34	230.14	26290.0	55693.0	-4.1399	.0000

Note. *n* for Principle = 244, *n* for Activity = 239.

Method section, only scores on the first half of the test were analyzed (that is, either part-whole and multiplication or number line and measuring items) for each student. This had the effect of reducing the number of cases by about half for each item.

Items given on the first day of posttesting constituted six subscales representing placing a fraction on a number line, labeling a mark on a number line, measuring, two types of area-partitioning problems (sharing pies and sharing brownies), and simple multiplication of fractions by whole numbers.

Table 15 reports descriptive statistics by group for all conceptual/ declarative measures.

As Tables 16 and 17 show, principle group students achieved significantly higher scores on measuring (M = 3.0606 compared to 2.5447 for the activity group) and on number line placement (M = 3.0606 compared to 2.5447 for the activity group). As noted earlier, ability to place fractions on a number line is generally held to be an indicator of quantitative understanding of fractions. Use of fraction notation in a measuring context demonstrates understanding of one application of the fraction symbol.

Table 15

Conceptual/Declarative Knowledge Means and Standard Deviations by Type of Instruction

Variable	Principle			Activity			Total items
	Mean	*SD*	*N*	Mean	*SD*	*N*	
SHARING PIES	1.1382	.7051	123	1.3386	.6071	127	2
SHARING BROWNIES	.2377	.4820	123	.2441	.5151	127	2
FRACTION OF NUMBER	1.9919	1.7993	123	2.4016	2.0674	127	6
NUMBER LINE PLACE	.8258	.9205	132	.5610	.8312	123	3
NUMBER LINE LABEL	1.1061	1.4102	132	.9837	1.4199	123	5
MEASURING	3.0606	1.5470	132	2.5447	1.6208	123	6
PARTS	1.5244	1.3870	246	1.4202	1.4022	238	4
FRACTION PICTURES	.1748	.3805	246	.2353	.4251	238	1
SLICES	.9472	.6890	246	.9538	.6766	238	1
HEIGHTS	.6951	.8669	246	.3277	.6638	238	1
LINES	.8659	.9003	246	.2899	.5989	238	1

Table 16

Multivariate Tests of Significance for Group Effect on Number Line and Measurement Tasks

Test name	Value	Exact *F*	Hypoth. *DF*	Error *DF*	Sig. of *F*
Pillais	.37067	3.33764	3.00	17.00	.044

Variable	Univariate statistics					
	Hypoth. SS	Error SS	Hypoth. MS	Error MS	*F*	Sig. of *F*
NUM. LINE PLACE	5.05475	14.82352	5.05475	.78019	6.47892	.020
NUM. LINE LABEL	.82940	33.84036	.82940	1.78107	.46568	.503
MEASURING	14.21580	47.30268	14.21580	2.48961	5.71004	.027

Note. *DF* = 1, 19.

Table 17

Multivariate Tests of Significance for Group Effect on Area and Multiplication Tasks

Test name	Value	Exact *F*	Hypoth. *DF*	Error *DF*	Sig. of *F*
Pillais	.23151	1.70712	3.00	17.00	.203

	Univariate statistic					
Variable	Hypoth. SS	Error SS	Hypoth. MS	Error MS	*F*	Sig. of *F*
SHARING PIES	1.98521	12.74956	1.98521	0.67103	2.95845	0.102
SHARING BROWNIES	0.08283	5.61894	0.08283	0.29573	0.28007	0.603
FRACTION OF NUMBER	9.29521	84.42585	9.29521	4.44347	2.09188	0.164

Note. *DF* = 1, 19.

Table 18 discloses MANOVA results for conceptual/declarative tasks given on the second day of posttesting. Here there is a significant overall group difference deriving primarily from the superior performance of the principle group on the use of fractions to compare heights and the lengths of lines (HEIGHTS and LINES); means for the principle group were .6951 and .8659 on these variables, compared respectively with .3277 and .2899 for the activity group.

Table 18

Multivariate Tests of Significance for Group Effect on General Conceptual/ Declarative Tasks

Test name	Value	Exact *F*	Hypoth. *DF*	Error *DF*	Sig. of *F*
Pillais	0.70307	7.10348	5.00	15.00	0.001

	Univariate statistics					
Variable	Hypoth. SS	Error SS	Hypoth. MS	Error MS	*F*	Sig. of *F*
PARTS	0.07367	95.61474	0.07367	5.03235	0.01464	0.905
FRACTION PICTURES	0.50330	5.02648	0.50330	0.26455	1.90248	0.184
SLICES	0.06351	17.19857	0.06351	0.90519	0.07016	0.794
HEIGHTS	15.10990	42.84698	15.10990	2.25510	6.70031	0.018
LINES	34.49222	22.18586	34.49222	1.16768	29.53918	0.000

Note. *DF* = 1, 19.

There were no significant differences on tasks involving: (a) relations between fractions and whole numbers (PARTS); (b) generating different representations for 3/6 (FRACTION PICTURES); and (c) writing fractions representing shares of partitioned pizzas (SLICES). Across the two days, the principle group did better on tasks reflecting quantitative knowledge about fractions, and equally well on sharing and other fraction situations presented to the activity group.

Pretest to posttest gains. There were four conceptual/declarative knowledge items, 1a to 2b, that were identical in type to four items on the pretest, items 16a to 17b (see Appendix B). In these items students were asked to shade in parts of circles and rectangles to represent "shares" of pizzas and brownies. They were also asked to write fractions identifying the shaded-in areas. Posttest and pretest scores on this set of items (PRETEST AREA and POSTTEST AREA) were analyzed in a hierarchical MANOVA with time (pretest-posttest) as a within-subjects factor, instruction group as a between-subjects factor, and classes nested within groups. Table 19 shows means and standard deviations for the pre- and posttest measures. There was a significant effect for time, $F(1,20) = 96.29$, $p < .001$, and no group by time interaction, suggesting that instruction improved knowledge about area representations for both groups. The relatively low numbers of cases in this analysis are explained in the discussion of conceptual/declarative knowledge results later in this section.

Computation results. Activity students had one or more class periods of practice figuring out numerators and denominators of equivalent fractions, and spent a similar amount of time adding and subtracting fractions. This led to the prediction that these students would do better than principle group students on the computation posttest, but a hierarchical MANOVA revealed no significant

Table 19

Means and Standard Deviations for Pretest and Posttest Area Measures, by Group

Variable	Principle		Activity	
	Mean	*SD*	Mean	*SD*
PRETEST AREA	.6250	.9639	.5169	.7705
POSTTEST AREA	1.3984	.8634	1.5593	.7343

Note. Principle n = 128. Activity n = 118.

difference. Mean, standard deviation, and *n* for the principle group were 5.79, 2.93, and 261; for the activity group, 6.68, 2.86, and 249. It seems plausible that both groups gained in computation knowledge, given time spent on it by the activity group, but this possibility was not systematically examined in this study, as the focus was on principled understanding, not computation.

Other Evidence Bearing on Task Validity

Evidence reviewed in this section relates to (a) the degree to which the assessments provide useful diagnostic information, (b) the extent to which results are consistent with those obtained in previous studies, and (c) predictable consistencies across tasks. Data on representational knowledge are discussed first. Means and standard deviations for each representational item type appear in Table 20.

Table 20
Representational Knowledge Item Means

Variable	Mean	SD
CIRCLE	4.29	0.95
CIRCLE EQUIVALENT	2.08	1.55
NUMBER LINE	1.20	1.39
NUMBER LINE EQUIVALENT	1.03	1.30
RECTANGLE	4.12	1.06
RECTANGLE-ALTERNATE SHADING	2.58	1.48
RECTANGLE EQUIVALENT	2.15	1.50
SEGMENT	3.32	1.23
SEGMENT EQUIVALENT	2.21	1.48
SET	3.88	1.10
SET EQUIVALENT	1.92	1.49
CIRCLE DISTRACTOR	3.83	1.08
NUMBER LINE DISTRACTOR 1	3.53	0.97
NUMBER LINE DISTRACTOR 2	3.27	1.32
RECTANGLE DISTRACTOR 1	4.02	1.02
SEGMENT DISTRACTOR 1	4.34	0.85
SEGMENT DISTRACTOR 2	4.27	0.87
SEGMENT DISTRACTOR 3	4.39	0.88

Note. Each variable comprises five items. "EQUIVALENT" indicates representations of equivalent fractions. "DISTRACTOR" indicates incorrect representations. $n = 515$ for all items.

Each variable in Table 20 represents a total score from five items distributed across five fraction types. As expected, the familiar part-whole representations (circles, rectangles, sets) were much easier than the less familiar number lines, with line segments falling in between. Simple or direct representations, in which the number of partitions matched the numbers used in the symbolic representation, were easier than equivalent representations, where the number of partitions did not match the numbers in the symbol. Figure 4 shows examples of direct and equivalent representations for the fraction 1/2.

Factor analysis of representational knowledge items. An exploratory factor analysis was performed, using principal components analysis for extraction, and varimax rotation. The purpose was to discover whether performance on the 18 representational knowledge types could be explained in terms of fewer underlying dimensions, and whether those dimensions could be plausibly interpreted in terms of item characteristics that students responded to in more or less similar ways. Results appear in Table 21. Factors with eigenvalues greater than 1 were accepted and the five factors accepted accounted for 70.2% of the variance in the original variables. Percentages of variance accounted for by factors 1–5 respectively were: 29.7, 16.6, 10.2, 7.7, and 5.9.

Different types of representations clearly loaded on different factors. Factor 1 has high loadings for equivalent part-whole representations of fractions; factor 2 has high loadings for simple part-whole representations and one type of segment distractor; and factors 3, 4, and 5 have high loadings for part-whole distractors, number line representations, and number line distractors, respectively. The factors appear to be orthogonal.

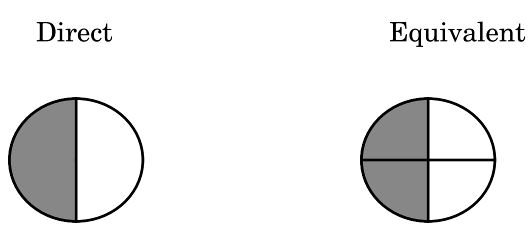


Figure 4. Direct and equivalent representations for the fraction 1/2.

Table 21

Rotated Factor Matrix for Representational Knowledge Items

Variable	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
CIRCLE EQUIVALENT	.90966	.08603	.10290	.12030	-.04623
RECTANGLE EQUIVALENT	.90237	.04796	.09233	.17181	-.09721
SET EQUIVALENT	.88101	.02827	.11131	.07347	-.07786
SEGMENT EQUIVALENT	.81557	.04533	.14367	.11211	-.08971
RECTANGLE-ALT. SHADING	.78228	.06539	.08996	.30630	-.23125
RECTANGLE	.02108	.76313	.11244	.18573	-.06919
CIRCLE	.13137	.70226	.01233	.29978	-.17517
SET	-.01541	.69796	.17351	-.15642	.12655
SEGMENT	.08018	.68158	-.03080	-.16694	.24825
SEGMENT DISTRACTOR 1	.04222	.65921	.27839	-.22216	.16083
RECTANGLE DISTRACTOR	.08626	.12403	.83199	.02157	-.00400
SEGMENT DISTRACTOR 2	.14066	.20376	.80108	.00865	-.00288
CIRCLE DISTRACTOR	.10079	.05945	.78977	-.08034	-.04062
SEGMENT DISTRACTOR 3	.40616	.04707	.48985	.35110	-.23471
NUMBER LINE	.22010	-.05501	.01211	.86344	.06529
NUM. LINE EQUIVALENT	.32617	-.02660	-.04158	.84420	.07719
NUM. LINE DISTRACTOR 1	-.07781	.09486	-.06547	.07198	.80376
NUM. LINE DISTRACTOR 2	-.30013	.10084	-.00896	.02748	.77900

Distractor effect. It appears to have been easier for students to avoid distractors than to select correct representations in the more difficult categories. For example, scores on NUMBER LINE DISTRACTOR 1 (M = 3.53) and NUMBER LINE DISTRACTOR 2 (M = 3.27) were more than twice as high as scores on correct number line representations, NUMBER LINE (M = 1.20) and NUMBER LINE EQUIVALENT (M = 1.03). For more familiar representations, such as circles and rectangles, there was little difference between means for correct representations and distractors. The overall mean for RECTANGLE, for example, was 4.12, compared with a mean of 4.02 for RECTANGLE DISTRACTOR.

Fraction difficulty. Predictions about fraction difficulty were also tested. Based on studies reviewed earlier, the expected easy-to-hard sequence was: 1/2, 2/4, 2/3, 4/6, 3/2. 2/3 turned out to be slightly easier than 2/4, but this difference was not significant. Otherwise, results supported the predicted hierarchy. Consistent with every reported study of fraction performance at this grade level, 1/2 is the easiest fraction for students to recognize and use. In the case of 2/3 versus 2/4, the smaller number of partitions in 2/3 representations may have compensated for the relative lack of familiarity of this fraction. Representations of 2/4 were significantly easier to identify than representations of 4/6, and 4/6 was easier than 3/2.

Table 22 contains means and standard deviations for total scores on each of the five fractions, and Table 23 summarizes results of the repeated measures analysis comparing contiguous pairs of measures in the difficulty sequence (the sequence in this case was based on the means obtained).

Table 22

Mean Number of Correct Representations by Fraction

Fraction	Mean	*SD*
1/2	13.50	2.36
2/4	11.20	2.92
2/3	11.36	2.20
4/6	10.84	1.94
3/2	10.55	3.54

Note. n = 515 for all fractions.

Table 23

Results of Pairwise Comparisons of Fraction Difficulty

Variable	Hypoth. SS	Error SS	Hypoth. MS	Error MS	*F*	Sig. of *F*
T2	2345.24466	3163.75534	2345.24466	6.15517	381.02054	0.000
T3	14.02913	3468.97087	14.02913	6.74897	2.07871	0.150
T4	65.02718	3629.97282	65.02718	7.06220	9.20777	0.003
T5	44.27379	5630.72621	44.27379	10.95472	4.04153	0.045

Note. T2 compares 1/2 with 2/3; T3, 2/3 with 2/4; T4, 2/4 with 4/6; T5, 4/6 with 3/2. *DF* = 1, 514.

Correlations with external measures. Relationships between total representation and computation scores, essay scores, and external measures were also examined. Nonparametric correlations were computed for the essay scales. Tables 24 and 25 report the correlations. Correlations with all CTBS (Comprehensive Test of Basic Skills) subscales were significant and moderately high, although not as high as intercorrelations among the CTBS subscales, which ranged from .70 upwards. Teacher ratings did not correlate highly with the representation or computation totals, or the misconception essay dimension. Relatively low correlations for teacher ratings may reflect the lack of detailed information they had about students early in the school year.

Other data. Classroom observational data reported in Appendix G suggest that there were no major differences in the management of classroom activities across instruction groups. Categories measured included teacher actions and student responses and behaviors. At intervals during the class period, observers noted the degree of control and direction by the teacher, size of working groups,

Table 24

Correlations Between Fraction Measures and External Variables

	Representation total	Computation total
CTBSMC.A	.5373**	.4805**
CTBSMCMP	.4716**	.4579**
CTBSRC	.4242**	.4313**
CTBSRV	.4210**	.4058**
TRFRAC	.1917**	.1521**
TRMATH	.1854**	.1603**
TROVER	.1636**	.1400**
TRWRITE	.1592**	.1380**

Note. CTBSMC.A = CTBS Math Concepts and Applications. CTBSMCMP = CTBS Math Computation. CTBSRC = CTBS Reading Computation. CTBSRV = CTBS Reading Vocabulary. TRFRAC = Teacher Rating of Fraction Knowledge. TRMATH = Teacher Rating of Mathematics Knowledge. TROVER = Teacher Rating of General Academic Ability. TRWRITE = Teacher Rating of Writing Ability.

** p <.01 (2-tailed).

Table 25

Non-Parametric Correlations Between Student Background Variables and Essay Scores

	GICQ	PN	FACT	MIS	INT
CTBSMC.A[a]	.4625***	.3992***	.3147***	.2518***	.4511***
CTBSMCMP[b]	.3106***	.2750***	.2309***	.1181*	.3370***
CTBSRC[c]	.4916***	.3826***	.3319***	.2508***	.4771***
CTBSRV[c]	.4632***	.3764***	.3406***	.2182***	.4008***
TRFRAC[d]	.3830***	.2307***	.3106***	.0735	.3683***
TRMATH[d]	.3704***	.2288***	.2993***	.1014*	.3634***
TROVER[d]	.4114***	.2298***	.3674***	.1080*	.4013***
TRWRITE[d]	.4271***	.2129***	.3819***	.0927*	.4047***

Note. CTBSMC.A = CTBS Math Concepts and Applications. CTSBMCMP = CTBS Math Computation. CTBSRV = CTBS Reading Vocabulary. TRFRAC = Teacher Rating of Fraction Knowledge. TRMATH = Teacher Rating of Mathematics Knowledge. TROVER = Teacher Rating of General Academic Ability. TRWRITE = Teacher Rating of Writing Ability.

[a]n = 301. [b]n = 302. [c]n = 303. [d]n = 457.

* p <.05. ** p <.01 (2-tailed). *** p = .00.

types and qualities of student responses, degree of student focus and appropriateness of behavior, and resources used. Differences in frequencies across groups were generally small, although principle group teachers did ask a higher percentage of questions and use more teacher-made materials.

DISCUSSION

Construct Validation of the Explanation and Knowledge Representation Measures

The success of this study depended to a great extent on the effectiveness of the instruction, which was given with the intention to test the instructional sensitivity of the explanation and knowledge representation measures and to increase variance on all measures. This was not essentially a study of instructional methods, but instructional effects constituted one of the linchpins for the validation effort. As the data collected formed a network of related evidence, validation information was obtained on various elements of the network. In construct validation, Cronbach and Meehl (1967) observe, "We examine the relation between the total network of theory and observations. The system involves propositions relating test to construct, construct to other constructs, and finally relating some of these constructs to observables" (p. 69). Traditionally, in construct validation studies "the proposition claiming to interpret the test has been set apart as the hypothesis being tested, but actually the evidence is significant for all parts of the chain. If the prediction is not confirmed, any link in the claim may be wrong" (p. 69). If predictions are supported, on the other hand, all elements of the chain are supported; this follows from the concept of a nomological network.

"The basic notion of nomological validity," argues Messick, "is that the theory of the construct being measured provides a rational basis for deriving empirically testable links between the test scores and measures of other constructs. It is not that a proven theory serves to validate the test, or vice versa. Rather, the test gains credence to the extent that score consistencies reflect theoretical implications of the construct, while the construct theory gains credence to the extent that test data jibe with its predictions" (1984, p. 48). In this study, a theory of the construct was used to design assessments and instruction and to make predictions. There were two major aspects of the construct theory:

1. A description of the structure of conceptual understanding in terms of relations among three types of elements: a symbol system, concepts and operations represented by the symbol system, and situations that give meaning to the symbols and concepts. This description provided a general framework for developing tasks and validity evidence. Relations that were assessed included the relations children discerned among different fractional representations of the same and different rational numbers (representational and conceptual/declarative knowledge tasks); relations between symbols and graphic representations (representational and conceptual/declarative knowledge tasks); relations between concepts, principles and symbolic problem-solving procedures (conceptual/declarative knowledge tasks, problem-solving tasks, and justifications); and relations between explicit (linguistically stateable) principles and operations, graphic representations, situational knowledge, and fraction symbolism (explanation task).

2. An analysis of the referential semantics of fractions, deriving from analyses by Ohlsson (1987, 1988) and others. This semantic analysis provided specification of the meaning of fractions as quantities and a set of related situations, for example, those in which quantities are measured and compared.

This construct theory guided the design of new instruction and assessments as well as predictions about the effects of instruction. Hypotheses about relations between instruction and outcomes were empirically supported. Hypotheses about relations among outcome and other measures were also supported. If the explanation and representational knowledge measures had not effectively discriminated students who received principled instruction from those who did not, the utility of these measures for instructional purposes would be impugned. If students who demonstrated high performance on explanation and representational knowledge measures had been markedly less successful on related measures, the validity of the explanation and representational knowledge scores would become more doubtful. If patterns of relations among representation item types had not been explainable in terms of structural differences in the representations, if students had circled items randomly, or if factors extracted in factor analysis had been uninterpretable, then the representational knowledge task could not be supported as a measure of fraction understanding. Given the effectiveness of instruction and the fact that the assessments operated as predicted, there is support not only for the construct theory but for the integrated assessment and instruction strategies derived from it.

Effects of Principled Instruction

The effort in this study to assess the relational structure of knowledge is consistent with a broad swatch of cognitive research. Studies comparing expert and novice performance have consistently shown that for many domains, the organization of knowledge typifies expert knowledge, and that organization around principles and structure tends to result in more flexible, generalizable knowledge (Chi & Ceci, 1987; Larkin, McDermott, Simon, & Simon, 1980; Silver, 1981). Novices tend to organize knowledge in superficial ways, classifying word problems, for example, on the basis of key words or the particular situations represented, rather than mathematical structure (Silver, 1981). Initial fraction instruction has likely reinforced the tendency to of young students to focus on the surface features of fraction representations. To counteract this tendency, an analysis was conducted to discover the “deep structure” of fractions, and that structure was used as the basis to design tasks from which its construction by students might then be inferred (Davydov & Tsvetkovich, 1991; McLellan & Dewey, 1895).

One of the most important results is that students receiving instruction on fraction principles expressed a higher level of principled understanding, and fewer misconceptions, on several different assessments. The principle group had superior performance to the activity group on the principles and misconceptions dimensions of explanation, as well as on other tasks requiring understanding of fractions as quantities. The latter tasks included recognition and generation of quantitative representations (number lines) in knowledge representation and problem-solving situations; placing fractions on number lines; and recording fractional measurements. This convergence of evidence supports the reliability of the conclusion that principled instruction had an effect; the same evidence supports the likelihood that these multiple tasks were measuring principled or conceptual understanding developed in response to instruction.

It was not expected that the activity group would receive no benefit from instruction, because the textbook-based activities were judged to be well-designed for their purposes and were strongly endorsed by activity group teachers. Certain assessments were therefore introduced to check whether the activity group did in fact profit from instruction. These assessments, which included the FACT explanation scale, computation test, and some conceptual/ declarative items, were keyed to the types of tasks included in activity instruction. On nearly all of these measures the performance of the principle group was equivalent to that of the activity group (the major exception being the FACT scale).

The achievement of the principle group is particularly impressive in light of the relatively small amount of time devoted to instruction on fraction principles. Of the seven days initially allotted for experimental instruction, two had to be redirected to teaching basic measurement principles rather than fractions, leaving only five days focused on principles related to the fraction concept. Instruction on part-whole representations did not improve performance on recognizing these representations to a greater extent than instruction on measurement principles did, nor did it produce significant differences in computation skill. But it did increase the likelihood that students would use part-whole representations in problem-solving and explanation situations. Some evidence, albeit limited, supported the possibility that both types of instruction may have influenced students' knowledge about part-whole representations.

Inadequate Overall Performance

Another finding that stands out is the low overall level of performance on complex tasks, which is consistent with results from many studies of fraction understanding. Students had moderate levels of representation knowledge, some of it probably developed in response to instruction during this study, but performance overall was not impressive. Mean representation scores ranged from 13.50 (out of 18 items) on the easiest fraction, 1/2, to 10.55 on the most difficult. But 98% of the students incorrectly circled at least one distractor. Only 10 out of 515 students avoided circling any distractors. Across the five fractions, students incorrectly circled an average of 7.13 distractors, indicating that many did not know (a) that the partitions shown had to be equal, or (b) how many partitions there should be. The most commonly circled distractors were number lines (M = 3.20), followed by line segments (M = 2.00), circles (M = 1.08), and rectangles (M = 1.02). Altogether there were 10 number line distractors, 15 line segment distractors, and 5 circle and 5 rectangle distractors. Students correctly identified fewer than half of the number line representations, and only about 63% of the area and set representations, which they had seen many times before. As reported above, however, the rate of success was higher than 80% on some item types, for example, circles and rectangles. But even with these representations, students were confused by equivalent representational forms and by incorrectly partitioned distractors.

With respect to the six problems, each of which has been taken as an indicator of fraction understanding in previous studies, 54% of the students successfully compared fractions of a pizza, but only 38% could solve an analogous problem involving fractions of lengths. Across the problems the percentage of students with correct solutions ranged from 9% on judging equivalence to 26% on finding a fraction between two mixed numbers.

On the explanation task, 25.6% of the students scored at the bottom of the GICQ scale, meaning that they expressed no fraction content knowledge. 63% of the students failed to give evidence of understanding any mathematical principle or concept, and 31% did not include any facts or procedures in their essays. Obviously the range of performance was restricted on most measures. This influenced the analyses that could be performed, and it would be useful for this reason to collect similar data from larger numbers of students who are well-instructed over a longer period of time. The main purpose of the assessments was not to rank students, however. It was to determine what kinds of fraction understanding students had achieved and to make that determination using methods consistent with construct theory and previous research, cited above. Among the many results consistent with previous research are those on fraction difficulty and those showing differentiated performance on number line and part-whole representations.

Time Limitations

Because of school district time constraints, the durations of teacher training and student instruction were unhappily short, so it was not expected that all or most students would undergo the kinds of profound conceptual change that might take a year or more under ideal conditions. As an application or interpretation of a rational number, a fraction must be conceived as the result of a relation between two numbers and at the same time as a single entity subject to mathematical operations. To grasp this point, even implicitly, requires considerable cognitive restructuring (Gelman et al., 1989) and a new understanding of mathematical units (Davydov & Tsvetkovich, 1991; Piaget et al., 1960). Based on belief that the meaning of a concept inheres in an elaborated cognitive structure, not in a simple definition, Menchinskaya (1969) estimated that it takes at least one-and-one-half years (assuming optimal instruction) to develop a concept such as "fraction," which is very close in kind to the "scientific" concepts discussed by Vygotsky. Understanding of such concepts develops over a long period of time, becoming deeper and more elaborated with increasing knowledge and experience.

In this study instruction and assessment focused on one particular concept, not on sampling from broad item domains varying enormously in content and infinitesimally in format, nor on vaguely defined constructs such as "problem-solving skills" or "higher level reasoning." It may seem that intensive analyses of all concepts in elementary mathematics would be impractical, but in fact, as Whitehead (1967) and others have argued, the number of essential ideas at any given level of school mathematics is limited. One might cull about seven or eight "big ideas" from a close reading of the NCTM (1989) *Standards* for fifth grade, for example. Unfortunately, analyses of textbook series have found as many as 250-

300 different important ideas introduced in a single year (Porter, 1989). Often new concepts are presented in 10 minutes or less, with little opportunity for discussion. In this study, slightly more than seven consecutive days were given to activities related to a single concept; this is an atypically-long block of time but still much too short for the major conceptual accommodations most students need to make. Nevertheless, the study provided a look at what the beginning stages of such instruction might be like, and how assessment might work to support it.

Implications of Constructivism

The fundamental purpose of mathematics education, it has been argued, is to initiate learners into a community of mathematical discourse (Greeno, 1991). Given this purpose coupled with constructivist epistemology, the need for improved assessments of students' understanding of and ability to use mathematical symbol systems is inescapable. From a constructivist's point of view, the purpose of instruction is to induce students to construct new understanding, using and building on what they already understand. Full realization of this aim depends on the availability of techniques for assessing understanding. In the ideal situation instruction is informed by frequent assessment of each student's understanding of mathematics as a system of knowledge, but instructional effectiveness will be attenuated when teachers do not have or do not use validated methods for discerning when systematic understanding has been achieved. In such instances instructional decisions have to be made without knowing whether students understand previously-taught concepts, or how they interpret topics under discussion.

In designing assessments for this study it was assumed that students who cannot use mathematical symbols appropriately and do not know the mathematical meanings of the symbols cannot be judged to know mathematics. Students who cannot effectively explain the meaning and justify the use of mathematical symbols, concepts, and operations are not yet full-fledged members of the community of discourse. And Greeno has argued that "a person's knowing of a conceptual domain is a set of abilities to understand, reason, and participate in discourse. . . . Critical components of these sets of practices include the appreciation and use of explanatory ideals that are shared within the community and provide basic modes and goals of explanatory discourse" (1991, p. 176).

Assessing Mathematics Understanding

Representational Knowledge

Based on constructivist ideas about how children move to greater knowledge and understanding, one would expect of children who had constructed mathematical meaning for the representations they had encountered in school that they would be able to see that there can be multiple ways to represent the same concept or mathematical structure; see that the structure of a task may be the same despite changes in representation (different representations do not necessarily imply a different concept or problem); use representations effectively in problem solving and be able to explain their use.

Representational fluency was used as another index for understanding of the fraction symbol. This strategy derived from the view that understanding of the fraction symbol can be viewed as generally analogous to understanding the written symbol for a word: greater understanding of a symbolic representation is closely related to the number of different synonymous meanings one can associate with the representation. To put this another way, the greater the network of meanings one can associate with a given symbol, the greater the conceptual understanding of that symbol. The number and variety of graphic representations students could associate with fraction symbols were therefore taken as indices of meanings for the fraction symbol, that is, as representations of possible meaningful situations, much as one might take synonyms for a word as representing extended meanings for that word. Another index consisted of a measure of the ability to generate and use representations in problem-solving justifications and explanations.

Explanations

Explanations are necessary to determine whether symbols and procedures have been memorized without understanding and whether students can connect symbolic activity to other knowledge. In addition, the NCTM *Standards* identify communication in mathematics as a central goal of instruction; communication is one of the core standards for elementary school. Another important reason for obtaining explanations is the mediating effect of linguistic ability on mathematical knowledge. Gelman and Lee (in press) review a number of recent studies in support of the conclusion that constructing explanations enhances learning; these studies examine the possibility that asking students for an explanation may help them organize their knowledge into a more explicit and more generalizable form. Other recent work suggests that linguistic representations of mathematics concepts, principles, and operations may mediate the construction of higher level principles (e.g., Gelman et al., 1989; Gelman & Greeno, 1989; Hiebert & Wearne, 1993). Gelman et al. (1989), for example, found a relation between "early mastery of the ordering of fractions and the ability to talk coherently about the principles underlying the construction of fractions." Lesh (1981), in a similar vein, has proposed that episodic experience that students gain from concrete materials may not provide retrievable knowledge without semantic information about the episodes and about relationships among different episodic experiences; his proposal buttresses the need for verbal interaction between learners and teachers and peers to compare episodic experiences and representations, and for teachers to assess students' understanding through their explanations. Referencing Thagard's theory of explanatory coherence, which proposes that coherence is a measure of the quality of an explanation, Gelman and Lee (in press) speculate that explanations elicited from students may provide insight into the coherence of their knowledge.

Implications for Educational Practice

Assessment

The NCTM and many others have argued that mathematical knowledge, because it is conceptually based, is not easy to evaluate. Nevertheless, if we wish to improve students' understanding of mathematical concepts, it is essential to know what it means to understand a given concept at different levels of complexity, and to be able to assess different levels of understanding. It is often difficult to tell on the basis of scores from multiple-choice tests whether students understand the concepts, principles, and methods within a domain (Nickerson, 1989). According to Linn et al. (1991), the psychometric methods used for selection, placement, classification, and certification may not be adequate for the design of diagnostic tests. It is not enough to show that a task discriminates between high and low achievers on some other measure, when one is interested, for example, in how students understand symbols or procedures, and how that understanding generalizes to novel problems and situations. Task development approaches based on psychometric methods are often indifferent to cognitive constructs like meaning or understanding, in the sense that they are not based on formal theories about them and their relation to performance. Understanding cannot be usefully measured by strategies deriving from theories in which it is not an important construct: behavioral psychology, for example, which dominated conceptions of learning in psychology and education in the 1960s. Thus advances in cognitive theory create the need for new kinds of tests that may be used as an integral part of instruction (Linn et al., 1991). The application of behaviorism in education has led to an emphasis on discrete skills defined as "precise, well-delimited behaviors" (Cole, 1990, p. 2). Many forms of criterion-referenced and objectives-based testing reflect this emphasis. Generalizability or transfer of skills and knowledge is not a central concern in such testing.

Evidence collected in this study supports the potential utility of representational knowledge, problem solving, justification, and explanation tasks in providing information about students' mathematical understanding. When prompted to do so, a majority of students were able to generate representations to support their conceptual explanations and problem-solving justifications. Students who were taught novel types of representations attempted to use those representations in their justifications and explanations. Students with higher levels of representational knowledge produced superior justifications and explanations. The types of principled knowledge measured by the assessments turned out to be teachable. And it was found that mathematics justifications and explanations could be reliably scored. In this study the prompt was open-ended—students were not told which representations to draw—but one could easily design a similar task to assess the ability to generate particular representations.

Results from this study also demonstrate the need to assess representational knowledge across a broad array of representations, and this array should encompass common misconceptions. On the evidence, the representational knowledge assessment strategy used here can provide useful information in a relatively efficient way on students' understanding of representational principles and misconceptions. Results from factor analyses of the representational items were interpretable in terms of specific categories of representations that students treated in more or less similar ways. Factors mapped onto specific item types. The number and types of factors extracted imply that most students had not constructed the principled understanding that would enable them to recognize (and generate) virtually any correct representation: Their performance was unstable across representation types. Also, there was a consistent fall-off in performance across the five different fractions. The knowledge that enabled students to recognize representations of one-half could not be generalized to other fractions.

Tasks developed and tested in this study should have broad applicability across diverse conceptual fields in arithmetic and higher mathematics. With respect to assessing representational knowledge, in many areas of mathematics, concepts are so closely tied to their representations that it is difficult to imagine the concepts without the representations, heightening the importance of knowing how students understand representations, and also the importance of choosing correct representations for assessment and instruction. It is easy to think of concepts and principles, for example, number, rational number, signed numbers, arithmetic principles, function, limit, and so on, that could be usefully assessed in terms of representational knowledge and the ability to talk about and use both representations and concepts.

The NCTM's (1989, 1993) recommendation of multiple sources of evidence on student understanding was taken up here in the form of several major tasks: knowledge representation, conceptual understanding, problem-solving justification, and explanation. When multiple assessments are used, inferences about understanding should be based on the convergence of evidence from different sources. Convergence takes on particular importance when one is interested in making inferences about a system of knowledge. Confidence in inferences about a student's understanding, that is, construct validity, will derive not simply from "external validity" evidence such as correlations of particular items with standardized achievement measures, nor from traditional measures of reliability, for example, item and scale reliabilities or rater reliability, but primarily from the patterns of relations among tasks representing different facets of domain understanding, and from the instructional sensitivity of the tasks.

Instruction

Overall, the study is thought to have particular relevance to educational approaches described by Prawat and Floden (1994) as "discourse oriented" and to the question of how assessment may serve the purposes of such approaches, variants of which have been demonstrated in different instructional contexts by Lampert (1986) and others. Validation of the assessments constructed for this study represents a step toward knowing how to assess students' understanding of and ability to use mathematical representations and language, which is an important index of their status as participants in the universe of mathematical discourse. One of the strikingly salient issues with regard to "discourse-oriented" teaching, and to closely related social-constructivist strategies, relates to the difficulty of choosing appropriate mathematical representations, that is, representations that help students construct mathematically correct ideas and that students at the same time find "authentic" and interesting. Choosing a representation is only a first step; one must also construct situations in which to introduce the representations in a way that gives students opportunities to learn about the represented concepts, not just the representation itself. And the activities chosen should minimize the possibility that students will focus on misleading or irrelevant aspects of the representations, as many students presently do with the fraction representations they encounter. A large percentage of students who took part in this study, for example, defined a fraction as something you take away from something else; for instance, "A fraction is a piece of pie that you eat." This is not necessarily a "wrong" definition—in some contexts it may be correct—but it seemed to be the only meaning many students could generate, thus limiting their ability to generalize fraction knowledge to a broad range of situations and problems,

Advocates of discourse-based instruction (e.g., Prawat & Floden, 1994) have begun to address some of its inherent difficulties, in particular the difficulty of helping students who are individually constructing mathematical meanings, based on their discourse activities, to arrive at mathematically acceptable interpretations of what they are doing. Overcoming these difficulties demands a higher level of knowledge and skill than many elementary school teachers have today. Some examples, albeit unrealistic in the face of current economic conditions, indicate how such knowledge and skill might be obtained. Japanese elementary school teachers, according to one report (Stevenson & Stigler, 1992), spend substantial amounts of time on curriculum issues, including time for regular peer meetings. Stevenson and Stigler report that entire meetings are sometimes devoted to the most effective ways to phrase a question about a topic or the best ways of engaging young children's interest in a lesson.

Teacher Judgment

Teachers seemed more willing than most researchers and theorists have been to accept *any* knowledge about fractions as evidence of understanding and to discount contradictory evidence. One teacher, referring to a student who wrote "A fraction is a (w)hole with parts," said, "I know students who write like this, and I know they understand fractions." But this teacher and others found it hard to say exactly how they knew that students understood fractions. On the evidence collected in this and other studies, most students have only partial, fragmented, and inconsistent knowledge about fractions.

There are a number of problems with the assessment stance taken by some teachers, which is that they can know whether students understand fractions without making any systematic assessment effort—that is, on the basis of their day-to-day interactions with students. For one thing, it is unlikely that teachers can assess understanding on this basis in classrooms where there is little discourse of any kind. And it cannot be taken for granted that students who participate in hands-on, small-group activities will automatically construct understanding. During pilot testing for this study, some students who were working very productively in small groups on brownie-and-cookie-sharing problems discovered an algorithm: "Put the number of brownies on the top and the number of people on the bottom and you get the right fraction." This algorithm was quickly communicated throughout the classroom and thereafter used by most students to solve the problems. The teacher observed that students who learned the algorithm from others did not seem to learn anything about fractions, even though they successfully completed all the hands-on and word problem tasks.

Another serious problem has been elucidated by developmental psychologists, who have argued that young children use preconceptual representations that are functionally equivalent but not identical to adult concepts (Kozulin, 1990; Vygotsky, 1978); for this reason, functionality cannot serve as a basis for differentiating conceptual from preconceptual thinking. Pseudo-concepts, as Vygotsky called them, look so similar to true concepts that adults often do not notice the difference, and the similarity of preconceptual to conceptual generalizations allows children to practice the use of conceptual generalizations before achieving awareness of the operations involved and the structure of the concept. "There is a great difference," Kozulin argues, "between an intelligent-looking action, and an adequate knowledge of one's own intellectual operations. A person may solve problems in a way suggestive of conceptual reasoning, but his or her own interpretation of this solution may still be carried out at the preconceptual level. With respect to educational practice this implies that there is a considerable difference between learning how to operate with concepts and becoming aware of the conceptual structure involved" (1990, p. 162). "At the center of this problem lies the relationship between symbol, concept, and the nonverbal referent," Kozulin concludes. "The success of pseudo conceptual reasoning hinges on the coincidence between symbol and referent with the concept out of picture." This is precisely the problem motivating this study.

Functionally appropriate use of fraction language in part-whole contexts cannot be taken as sufficient evidence of conceptual understanding, and it is hoped that advances in assessment methods will help make clearer for teachers and others the distinction between conceptual understanding and merely functional or instrumental use of mathematical language and representations. This highlights another reason for assessing explanations: to determine the level of abstractness or generality of concepts. On this point it is useful to recall the finding by developmental psychologists that children taught novel words show a proclivity to be influenced by the first context in which a word appears, and to have difficulty abstracting the meaning from this context (Werner & Kaplan, 1963; Vygotsky, 1978). Children tend to fuse the meaning of a word with the first sentence in which they hear it and to carry this meaning forward into future applications (Kozulin, 1990). Something like this seems to have happened with fractions, where children fuse the meaning of the term with an image of a colored-in piece of pie, or with a piece of pie removed or eaten, or with procedures for counting pieces of a pie. Even students who studied fractions in measurement situations for more than a week, showed a strong tendency to talk about fractions as parts of pies, and to draw pictures of "pies." We can expect that it will take a considerable amount of time and effort for students to overcome the limitations of part-whole representations, and changes in their explanations and representations can give us insight into the evolution of their understanding.

Assessment Purposes

Assessment tasks can elucidate for teachers and students what the essential math skills and knowledge are. The tasks and associated rubrics described here circumscribe a particular concept field in a particular way, making clearer what students ought to know and be able to do. Although the investigation focuses on fractions, and in fact on a particular type of fraction understanding and its relation to performance on problem-solving and explanation tasks, applications of the findings should not be limited to this domain. In principle, the method of building assessment on conceptual analysis should be applicable to other grade levels and other areas of mathematics, but this is research still to be done. What the long-term consequences of these assessments on instruction might be is also an empirical question, bearing on the consequential validity of the tasks.

There is no question that complications arise when one tries to achieve some integration of assessment for diagnostic and accountability purposes. Teachers and students need detailed diagnostic information, but at a system level one wants information on, among other things, the transfer and generalizability of student performance, or on the validity of inferences one can make about program quality. Assessment design in this study began with the aim of constructing tasks that might yield useful diagnostic information, in the hope that assessments designed and validated for classroom-level purposes could eventually also serve system-level assessment purposes. An experimental design with randomization was used in order to permit some degree of generalization about the sensitivity of the assessments to different types of instruction and about the effectiveness of the instructional interventions. The essential research has still to be done on how results from measures such as those studied here might be made useful for large-scale decision making and curriculum planning.

A broadening of the research in other directions would also be useful. Previous research and this study show that one can build successful assessment and instruction activities for children that reflect key mathematical constructs, and can assess at least some essential aspects of fraction understanding. At least one construct theory exists that can be used to build systematic instruction and assessment. What is not yet clear is how, given the complexities of rational number knowledge, one can design experiences that encompass and unify all aspects of that knowledge, as well as how one might extend and vary the task formats to include among other things hands-on performance assessments.

Prospects for Reform

Can new assessments and instruction be integrated to support the development of conceptual understanding? There are reasons to believe that they can, but to date there have been few concrete examples to show how this might be done. The study described here limns some of the possibilities, but these possibilities could not have been explored without the full and committed participation of teachers, principals, and school district administrators. Given the current interest in educational reform across the curriculum, it is distinctly propitious that 23 teachers, nearly half the fifth-grade teachers in the district, would agree to participate with such enthusiasm in a research study. To further the aims of the study, teachers in the principle group were willing to try out a new method for teaching fractions with only minimal training. Activity teachers were willing to resequence their normal curriculum. Teachers in both groups gave up at least two-and-a-half class periods for testing, knowing in many cases that their students would not do well. Teachers were paid only an hourly rate for two hours' meeting time. They participated, they said, because they wanted to learn more about new assessments. Their commitment to this collaboration with researchers is a sign of hope for mathematics education reform.

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Appendix A

Principle Instruction Guidelines
and
Student Activity Sheets

Lesson 1
Measuring with Non-standard Units

Activities:
• Use small objects to measure larger objects.
• Record measurement results.

Key Ideas:
• Any length might be used as a unit to measure longer lengths.
• There should no gaps or overlaps when a measuring tool is used to measure a length.
• The whole length must be measured.
• Lengths shorter than the measurement unit can be expressed as a fraction of the unit.
• The length of an object may be expressed in many different ways, depending on the units used to measure it.
• If you use a smaller unit to measure an object, the measurement outcome will be larger.

Sample Lesson

Materials:
1 rectangular book or 1 large book and 1 small book
several small objects: e.g., paper clip, pencil, small box

"Can we measure the blackboard with this book? How? Does someone want to try it?"

After a volunteer has been recruited to measure the board, the class should discuss whether the measurement procedure he or she uses is acceptable. If the volunteer measures correctly, propose some possible wrong methods, such as not starting at one end of the board, failing to measure the whole board, incorrectly laying down the measuring tool.

Once the measurement has been successfully carried out, write the result on the board. If there is something left over (i.e., if the book does not measure the board evenly) the result can be written like this: "The blackboard is between 12 and 13 math books long."

The class should also try to estimate the size of the "left-over" length as a fraction of the original unit, one book-length. The new result might be: "The blackboard is about 12 1/2 books long." Point out that this result is more precise than the original measurement.

Say: **"Whenever you use an object like this book to measure something else, the length of that object is called the unit of measure. In the measurement we just did, we could say the blackboard is about 12 or 13 units long."**

Now ask: **"What would happen if we used the width of this book, instead of its length, to measure the blackboard?"** Demonstrate how the book would be placed on the board. **"Would we get a larger or smaller answer than we did before? Why?"** --if the measuring unit is smaller, the measurement result is larger

"How long do think the board is, measured by the width of this book? Would someone do this measurement, so we can check?" Discuss and record results as above.

Lesson 1 continued

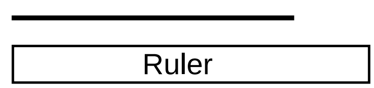
"You can see we never know for sure how long a line is. Can you help us out? Can you tell us a better way to measure? Take out Activity Sheet D. You can use this sheet to write instructions for us."

"Write down all the steps we need to do to measure a line correctly."

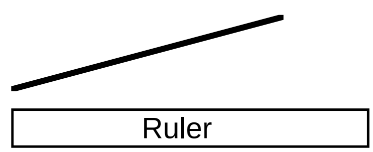
Give students sufficient time (2-10 minutes?) to record their ideas. You may need to assist some students by drawing a line to measure and telling them to write down what they do when they measure the line.

After students have written down their instructions, the whole group should discuss them. Elicit and write down ideas on the board one at a time, in each case asking whether other students have a similar instruction on their list. Eventually the list on the board might look something like this (exact wording is not important):

- Choose a measuring tool with the right units.
- Align the measuring tool with the line to be measured, like this:



not like this:



- Line up the 0 on the measuring tool with one end of the line to be measured.
- Read the number of units closest to the other end of the line.

Lesson 2
Measuring with Fractions

Activity:
• Estimating fractions of a unit length.

Key Idea:
• Lengths shorter than the measurement unit can be expressed as a fraction of the unit.

Sample Lesson

"Now let's look at what happens when you try to measure something that's smaller than the unit you're using. This happened to us in Activity 1. Sometimes we had leftover lengths that were smaller than the book we were using to measure. We estimated these lengths as fractions of the book, such as 1/2 or 1/3."

"When you're using a measuring tool, like this book, to measure something and there is some length left over, you can try to guess how many of those lengths it would take to equal the length of the book."

Draw a line segment about 1/2 of the book's length on the board. **"Let's say we want to estimate how long this line segment is, compared to this book. If it takes 2 of these lengths to equal the length of one book, how long is this line segment ?"** --1/2 the length of the book

Draw a line segment about 1/3 of the book's length on the board. **"How many of these lengths do you think it will take to equal the book? Then how long is this line segment, compared to the book?"** --1/3 the length of the book

Draw a line segment about 1/4 of the book's length on the board. **"How many of these lengths do you think it will take to equal the book? Then how long is this line segment, compared to the book?"** --1/4 the length of the book

"What if we had a line segment that was more than 1/2 the length of the book but not as long as the book? What are some fractions we could write? What are some fractions that are greater than 1/2 but less than 1 whole?" -2/3, 3/4, etc. (Possible question for advanced groups: **"How can you tell when a fraction is greater than 1/2?"**)

At this point it is not crucial that students understand the meaning of fractions such as 3/5, 7/9, etc., only that they have some experience estimating fractions in a measurement context.

Now have students work individually, in pairs, or in small groups on Activity Sheets A and B. Read the directions at the top of Activity Sheet A aloud. Students may cut out the gray measuring strips, but you should discuss first whether there is any way to do the activity without actually cutting out the strips.

Lesson 3
Standard Measuring Units

Activity:
• Using a ruler to measure line segments.

Key Idea:
• Why we use standard units and measuring tools.

Sample Lesson

"In the last few activities we've been using things like books and pencils to measure objects. People don't usually use these kinds of things for measuring. Why not?"
-- books, pencils, etc., aren't all the same size, so measurements would not be reliable; books are too heavy, and so forth.

"What kinds of things are used for measuring?" -- rulers, yardsticks, meter sticks, measuring cups, scales, etc. For each example offered, discuss what it measures and what the measurement units are.

Then have students take out and try Activity Sheet C.

Lesson 4
Measurement Principles

Activity:
• Explaining measurement procedures and principles.
• Recognizing "bad" rulers.

Key Ideas:
• Measurement principles described in earlier lessons.

Sample Lesson

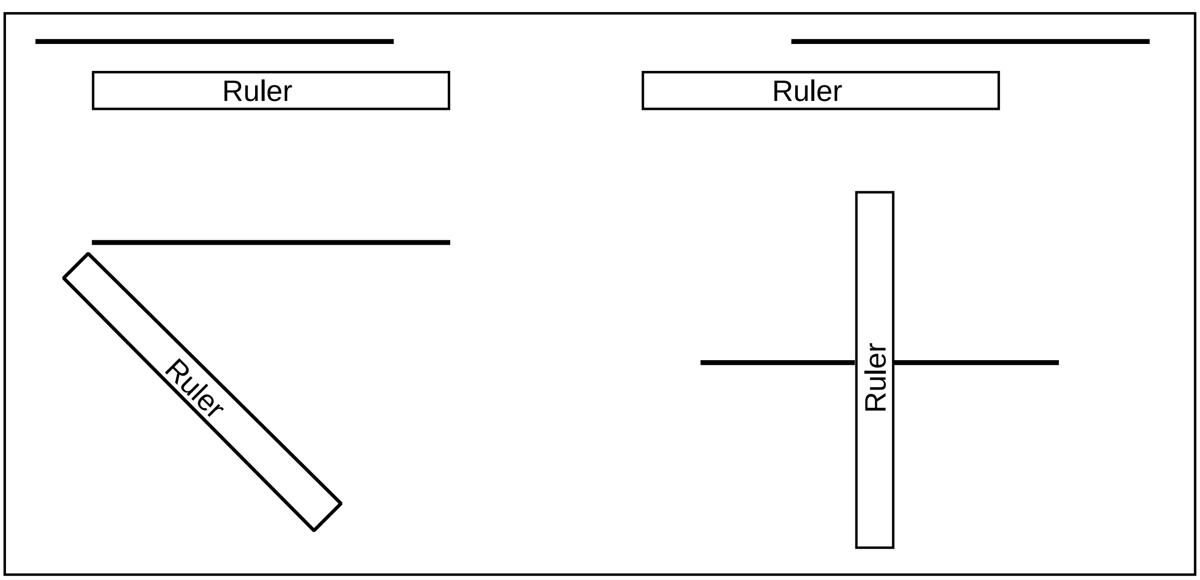
Materials:
You will need:
Ruler
Each student will need:
Activity Sheets D and E

Begin by saying: **"Imagine a person from another planet suddenly appeared in our classroom and said this:" (or you could say: "Imagine I'm someone who has just landed here from another planet.")**

"I want to learn about measuring. On my planet, we have a problem when we try to measure things. No one can ever agree on how long anything is. We can't tell how far away anything is either. I thought it was only going to take me 2 years to get to Earth, but it took me 200 years."

"Can you help us figure out what we're doing wrong? Here are some of the ways we measure line segments."

Draw a line segment about 12" long on the blackboard. Demonstrate several incorrect ways to measure the 12 inch segment you have drawn, such as those shown below. Record the results of each measurement, e.g., 10 inches, 5 inches, 11 inches, 6 1/2 inches etc. Ask students to suggest what might be wrong about each method.



Other rules may be offered, and the class should discuss and decide whether each new rule is necessary or not. The class should also discuss whether the list is complete and as clear as possible. The goal in this discussion is to make sure that all students understand and can explain how to use a ruler to measure lengths, and that what they know is stated as clearly and comprehensively as possible.

If students have trouble describing how to measure, you might want to demonstrate some of the wrong ways of measuring a line segment again, in each case asking: **"Will this work? Why or why not?"** Answers to these questions can be turned into measuring rules.

After all important instructions have been listed, students should add any they forgot to their own lists. Then collect all sheets.

Follow-up activity, or possible homework: Activity Sheet E. Read directions aloud and discuss them with the class. Students may complete the sheet in small groups or individually.

Lesson 5
Measuring and Comparing Lengths

Activities:
- Measuring two line segments.
- Writing a fraction to express the comparison between two measured quantiities.

Key Ideas:
- Fractions are numbers that can show the relation between two quantities (in this case, two lengths).
- If two lengths are measured by the same unit, either length can be expressed as a fraction of the other.
- The length of an object may be expressed in many different ways, depending on the units used to measure it.

Sample Lesson

Materials:
Ruler

Preparation:
Draw two line segments on the board, one 12" long, the other 24." Label them as shown below:

Begin by saying: **"Here's a new way to think about fractions. We can write a fraction to show how long line segment A is, compared to line segment B."**

"Is line segment A longer or shorter than segment B? How long do you think line segment A is, compared to segment B?" (Other possible questions: **"Can you guess what fraction it would be? Why do you think it's that fraction?"**)

"Let's write your guesses on the board." (After several guesses have been recorded, you may ask whether anyone else has a guess, then ask how many students agree with each guess and record the total. Try to elicit an explanation for each guess, and if possible elicit evaluations of the possible correctness of each guess.)

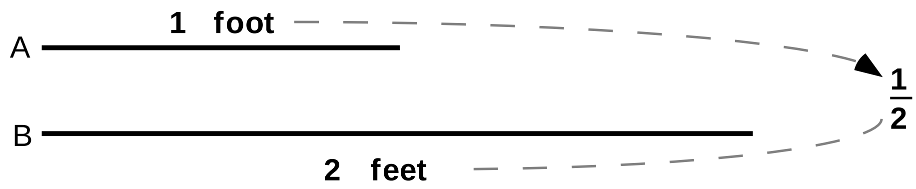
"When you guess, you're making an 'estimate.' 'Estimate' is another word for 'guess.' Estimates are usually not exact. We can get a more exact idea about how long one line segment is, compared to another segment, by measuring both segments."

"How can we measure these two line segments?" Recruit one or two volunteers to come to the board and measure the segments. The segments may be measured in feet or inches. Write the results on the board.

"Now that we have measured both line segments with the same units, we can write the length of line segment A as the top part of a fraction, and the length of line segment B as the bottom part."

Lesson 5 continued

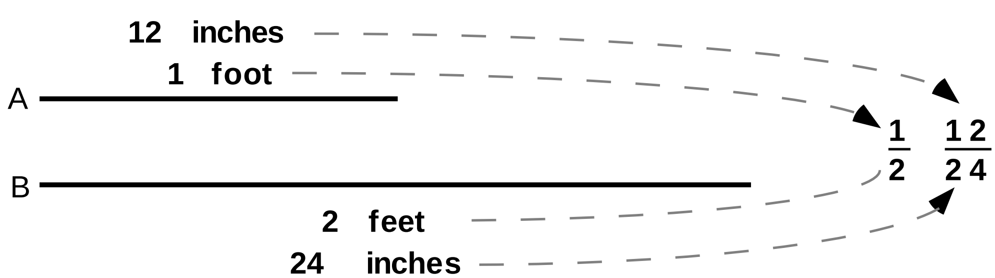
Add the labels shown below to your diagram on the board (assuming students have measured the line segments in feet; otherwise write the number of inches):



"This shows how long segment A is, compared to B. Line segment A is 1/2 the length of line segment B."

"These measurements use feet as the unit of measurement. Is there any other unit we could use to measure the segments?" (If there's no response, ask: **"What do these numbers on the ruler stand for?"** --inches.) **"Can we measure these segments in inches?"** Recruit volunteers.

If the line segments are measured in inches first, prompt students to try other units, e.g., feet. Ultimately both segments should be measured in both feet and inches, and you should have the following diagram on the board (students may also suggest other units, and these may be added to the diagram):



"The two fractions we've written show the same thing. Line segment A is 1/2 as long as line segment B. Or you could say line A is 12/24 as long as line B. Both fractions are the same."

Write on the board:

$$\frac{1}{2} = \frac{12}{24}$$

"1/2 and 12/24 are equivalent fractions."

This is a preliminary illustration. It is not necessary for students to understand the concept of equivalent fractions fully at this point.

Repeat this activity with several other pairs of line segments, such as those shown below. Be sure to include some instances where A is longer than B.

A	B
12"	36"
6"	12"
18"	12"
9"	6"

Discuss with students other quantities that might be compared using fractions, such as money, heights or weights of different persons, etc. Write some fractions on the board that illustrate these comparisons.

Students might want to try writing a fraction that shows how tall they are, compared to a 7 feet tall basketball player. The numerator and denominator of such fractions may be in feet or inches.

Be sure students understand that both quantities shown in a fraction must be measured in the same units: you can't show how large 6" is, compared to 1 foot, by writing 6/1 (this would be a ratio, not a fraction according to our definition).

Instructions for Student Activity Sheets F and G

Read directions for Activity Sheet F aloud. Work out the first problem or two with the class. For example, ask how many units long A and B are, as measured by the grid. If necessary, recall previous activities in which fractions were used to compare one line segment to another. Then have students complete sheets F and G individually or in small groups.

The final task is an assessment which may be completed individually or in small groups. Instruct students to draw two line segments on a sheet of paper. Each segment should be labeled in some way. Then students should measure both segments, record the results, and write a fraction expressing how long one segment is, compared to the other. They may use any available rulers or may cut out the rulers provided on the "Measuring Tools" sheet. This activity may be repeated if you wish.

Lesson 6
Partitioning Line Segments

Activities:
- Partitioning a line segment into equal-size segments.
- Labeling segments with fractions.

Key Ideas:
- To find a fraction of a line segment, the line segment must be partitioned into equal-sized intervals.
- A line segment can be partitioned into any number or equal pieces.
- The size of the pieces is inversely proportional to their number.
- No matter how small a line segment you have, you can always partition it into smaller segments.

Sample Lesson

Materials:
Yardstick or ruler

Preparation:
Draw a 36" line segment on the blackboard, starting on the left side of the board.

"Let's say this is a piece of gold wire. Some children are going to cut it up to make things. What could you make out of gold wire?"

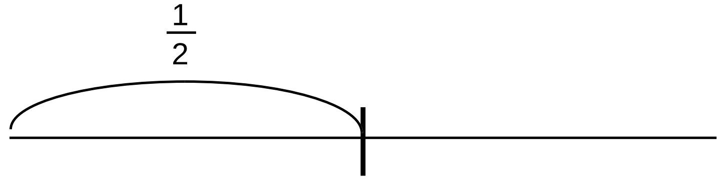
Alternatively, you could suggest that the line segment is a piece of yarn or string, or some similar object. You could even give students pieces of string or yarn to cut.

"Ok, let's say 2 children want to share this wire equally. Where should they cut the wire so each gets the same-sized piece? Can someone come up and mark where they should cut the wire?"

After a volunteer has marked where the wire should be cut, the class should discuss whether the two pieces are equal. If the class agrees the pieces are not equal, other volunteers should try to mark the line segment, until all or most students agree on where the cut should be made. Then proceed.

"If both of these pieces are equal, how much of the wire will one student get?" -- one half
"If you divide anything into two equal pieces, how much will each piece be, compared to the whole thing?" -- one half

Label one half of the line segment.



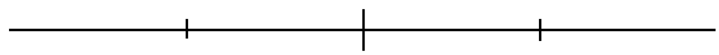
Note: The half-way mark needs to be large enough so that smaller marks can be added to the line segment, as shown on the next page.

Activities:
• Partitioning a line segment into equal-size segments.
• Comparing fractions.

Key Ideas:
• Equal-sized segments of a line segment can be expressed as fractions; e.g., if you divide a line segment into four equal segments, each segment is 1/4 the length of the original line segment.
• The size of the pieces is inversely proportional to their number.

"Now let's imagine that 2 more children come and all four children decide to share the wire. How should they cut the wire now?"

As before, volunteers should mark the line.



"If all four of these pieces are equal, how much of the wire will one student get?" -- one fourth

Label 1/4 of the line.

"If you divide anything into four equal pieces, how much will each piece be, compared to the whole thing?" -- one fourth

"If two people divide the wire, each gets 1/2. If four people divide it, each gets 1/4. Which is more, 1/4 or 1/2? Why?"

"Now what would happen if 4 more people came? That would be 8 people all together. How would they divide this wire?"

Mark and label the line segment as before. The diagram should now show 1/2, 1/4, and 1/8. Then ask students to compare several fractions, e.g.: **"Which is more, 1/8 or 1/4? 2/8 or 1/4? 2/4 or 1/2?"** It will be easier for students to make these comparisons if you label all fourth and eighth marks: 1/4, 2/4, 3/4, 1/8, 2/8, 3/8, etc. Discuss the fact that some marks can be labeled by two or three different fractions: these fractions are "equivalent"-- they have the same value.

"OK, let's draw a new gold wire. See how easy it is to make a gold wire?" Erase the old line and draw a new one. **"Now let's make up our own story. How many children could divide up this wire?"** --e.g., five

"How should they cut the wire? How much of this wire does one child get?"
-- one fifth
Repeat this with several other numbers.

Activity:
• Comparing fractions.

Key Idea:
• You can divide any line segment into as many equal pieces as you want.
• The size of the pieces is inversely proportional to their number.

"What's the largest number of equal pieces you could divide this wire into?"

Write guesses on the board. Point to some of the guesses, asking, **"If this many children (e.g., 500) divided up the wire, how much would one child get?"** --1/500 **"Can anyone write that fraction?"**

Keep asking for larger numbers until students realize that the number could be infinitely large. Students should also understand that as the number of pieces increases, the size of each piece decreases.

Write several fractions on the board and ask students which is the largest and which is the smallest. For example,

$$\frac{1}{6} \quad \frac{1}{5} \quad \frac{1}{7}$$

Repeat with other fractions until all students understand how to compare these "unit fractions."

Lesson 7
Fractions on Number Lines

Activities:
• Placing numbers on number lines.
• Discussing number line conventions.

Key Ideas:
• Each point on a measuring strip or number line represents a distance from a reference point, 0.
• Between any two numbers you can always find another number.

Sample Lesson

Draw a new line segment.
"Let's make this into a number line. It looks a lot like the gold wire, doesn't it? What should this segment have, to make it a number line?" ....numbers

"OK, how about this?"
Draw and discuss:

3 1 7 4 0

"Or how about this?"
Draw and discuss:

10 20 30 40 50

Draw some other wrong examples if you wish.
Students should recognize that the numbers should be ordered and equidistant. If students do not propose this in discussion, you may have to tell them that these are number line conventions.
"**On a number line, each number shows how far that point is from 0. 3 is three times farther from 0 than 1 is, and so on. Number lines also extend to infinity in both directions."**

Then draw and discuss the following:

0 10 20 30 40

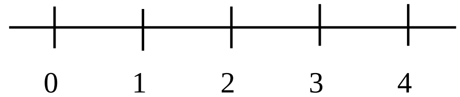
"On this number line, 20 is twice as far from 0 as 10 is. Why?"

"What numbers could you write between these numbers? For example, what might go between 10 and 20?"
Students should add numbers such as 17, 42, etc., to the number line, discussing the placement of each number.

Activities:
• Placing numbers on number lines.
• Discussing number line conventions.

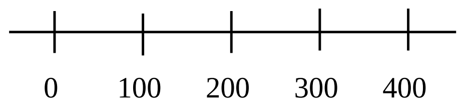
Key Ideas:
• Each point on a measuring strip or number line represents a distance from a reference point, 0.
• Between any two numbers you can always find another number.

"Here's another kind of number line: "

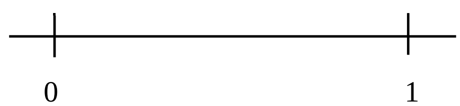


"What numbers could you write between these numbers?" To help students think about numbers such as 2 1/2 and 3 1/4, you might talk about ages: **"How old is someone who's more than 2 but not yet 3 years old? Where would we put that on the number line?"**

Also ask, **"If these numbers represented dollars, where would $2.00, $2.50, $.50, $3.75, etc., go on this number line?"** Note: Cents must be converted to fractions of a dollar. If students try to write a number like 75 on the number line, remind them that the numbers show dollars. You could draw a separate number line showing cents and have students place numbers like 275 on it:



"Next let's talk about this number line:"



"Are there numbers between 0 and 1? What are some examples?"

Draw guesses on the board. Discuss whether each guess is sensible. If there are no guesses refer to earlier discussions about dividing up a wire.

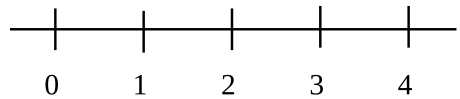
"How many numbers are there between 0 and 1?" Try to elicit the possiblity that any line segment can be divided into any number of pieces--between any two numbers, there are an infinite number of numbers.

Lesson 7 continued - 2

Activity:
• Placing numbers on number lines.

Key Ideas:
• Understanding how to represent fractions and mixed numbers on a number line.

Draw the following on the board:



Then discuss this question with the group: "**How many numbers are there between 3 and 4?"** (Fractions and mixed numbers are numbers.)
Write several guesses on the board.

Have students mark some mixed numbers between 3 and 4 and have the class discuss which numbers are closer to 4 and which are closer to 3. Students should explain their answers.

It might be useful during this discussion to place 1/2-unit and 1/4-unit marks along the number line and discuss the fact that numbers like 3 1/2 can be expressed as 7/2 or 14/4.

"Now we'll work with partners (or in small groups) to see if you can find some fractions."
Hand out Activity Sheet H (graph paper).

Write this problem on the board: **"Find a fraction between 0 and 1/2. Draw a number line on the Activity Sheet to show your answer."**
Students sitting next to each other may work in pairs, or you may want to create other groupings. After students work for several minutes on the problem, the whole group should discuss solutions that have been found.

Then try another problem in the same way: **"Find a fraction between 1/2 and 1. Make a picture to show what you do."**

Also have students try problems like these:
"Find a fraction between 5 and 6."
"Show hwere 2 1/2 pizzas would go on a number line."
"Show $5.50 on a number line."
"Show where 3 1/4 feet would go on a number line."
Explain to students that measuring units are usually not shown on a number line, only numbers.

Write all problems on the board. Students may use Activity Sheets I and J to draw additional number lines if necessary.

Lesson 8
Fraction Equivalence

Activities:
• Fraction Game from Debbie Brewer.
• Identifying equivalent fractions.

Key Ideas:
• Equivalence of fractions.

Fraction Game

Materials:
5 different-colored paper strips for each student.
Note: Debbie used 18" by 4 1/2" strips of colored construction paper, but white 14" x 1 1/2" strips will also work. If you don't have construction paper, 8 1/2" x 14" sheets of paper are enclosed; students may cut fraction strips and game boards from these sheets.
Scissors, felt pens, wooden cubes labeled with fractions.

Preparation:
One strip will serve as the game board.
Other strips should be cut as follows:
Ask each student to hold up a strip and show how to fold it in half. If the folds are correct, students should cut on the fold and label each piece 1/2. Do this with each strip so students have 1/4s, 1/8s, and 1/16s.

Label the six sides of the cube 1/2, 1/4, 1/8, 1/8, 1/16, 1/16.

Object of Game:
Be the first to cover the game board.

Playing the Game:
Each student has a game board.
Students take turns rolling the cube, putting the correct fraction piece on their own game board after each roll.

Related Discussion:
Either before or after playing the game, have students use their fraction pieces to answer questions you ask, such as: **"How many sixteenths equal one whole? How many fourths equal 1/2? How many eighths equal one-fourth, etc.?"**

Then have students work on Activity Sheet K individually or in small groups.

Lesson 9
Equivalent Fractions

Activities:
• Recognizing and finding equivalent fractions
• Illustrating equivalent fractions

Key Ideas:
• The fundamental property of fractions:
- If the numerator and denominator of a fraction are multiplied by the same number, a new fraction is generated that is equivalent or equal in value to the original fraction.

Possible Lessons:

You might introduce the idea of equivalent fractions by asking questions like this:
"Which would you rather have, 1/2 of a dollar, or 2/4 of a dollar? Why?"
"Which would you rather have, 2/3 of a pizza, or 4/6 of a pizza? Why?"

The goal is for students to discuss the idea that 2 fractions can have the same value, even though they do not look the same.

After the idea of equivalence has been introduced, students can try Activity Sheets L, M, N, and O. It might be helpful to make overheads of sheets M, N, and O. The directions, examples, and first question or two can be covered by the whole class, then students can complete the remaining questions individually or in small groups.

Lesson 10
Adding Fractions

Activities:
- Adding fractions
- Placing fractions on number lines

Key Ideas:
- Representing fraction addition on a number line.

Possible Lessons:

You might introduce the idea of fraction addition by asking questions like this:

"If you had 1/2 of one dollar, and someone gave you another half dollar, how much would you have altogether?"

"We can show what happened in this situation by drawing a number line like this."
Draw a diagram similar to the left of the diagram at the top of Student Activity Sheet P.

After you discuss other examples of fraction addition, students can try Activity Sheet P and Q. It might be helpful to make overheads of these sheets. The directions, examples, and first question or two can be covered by the whole class, then students can complete the remaining questions individually or in small groups. Sheet R (graph paper) may be used to draw more number lines to illustrate addition problems, or if you wish, to give additional practice in placing fractions on number lines.

Student Activity Sheets

NAME: ____ **New Student Activity Sheet A**

Measurement Units

Most of the time we measure lengths in units such as feet, inches, meters, or centimeters. These are standard units of measurement, but any length can be used as a unit to measure other lengths.

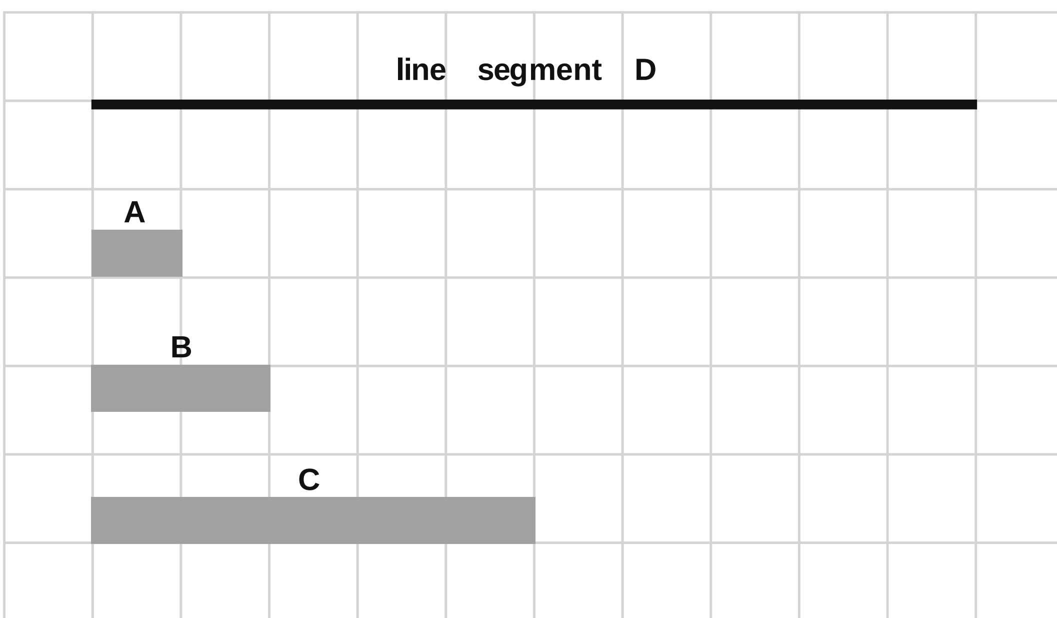
Below you see 3 different strips of gray paper (A, B, and C) that could be used to measure the length of line segment D. When you use one of the strips to measure, the length of that strip is "1 unit."

Answer these questions:

If the length of strip A is 1 measuring unit, how long is line segment D?

If the length of strip B is 1 measuring unit, how long is line segment D?

If the length of strip C is 1 measuring unit, how long is line segment D?



NAME: ______________________________ New Student Activity Sheet B

Units of Measurement

Answer each of the questions below.

If the length of strip A is 1 unit, how long is line segment G?

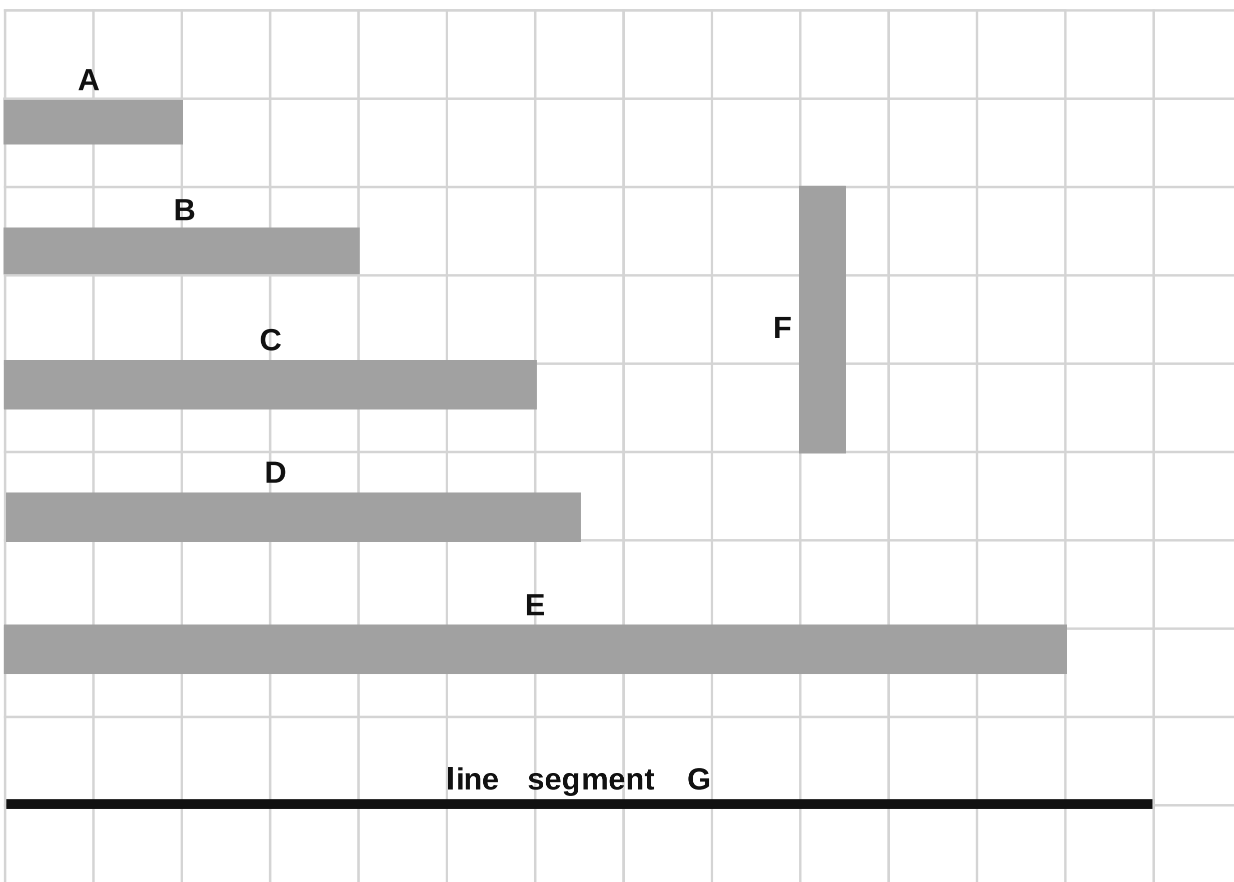
If the length of strip B is 1 unit, how long is line segment G?

If the length of strip C is 1 unit, how long is line segment G?

If the length of strip D is 1 unit, how long is line segment G?

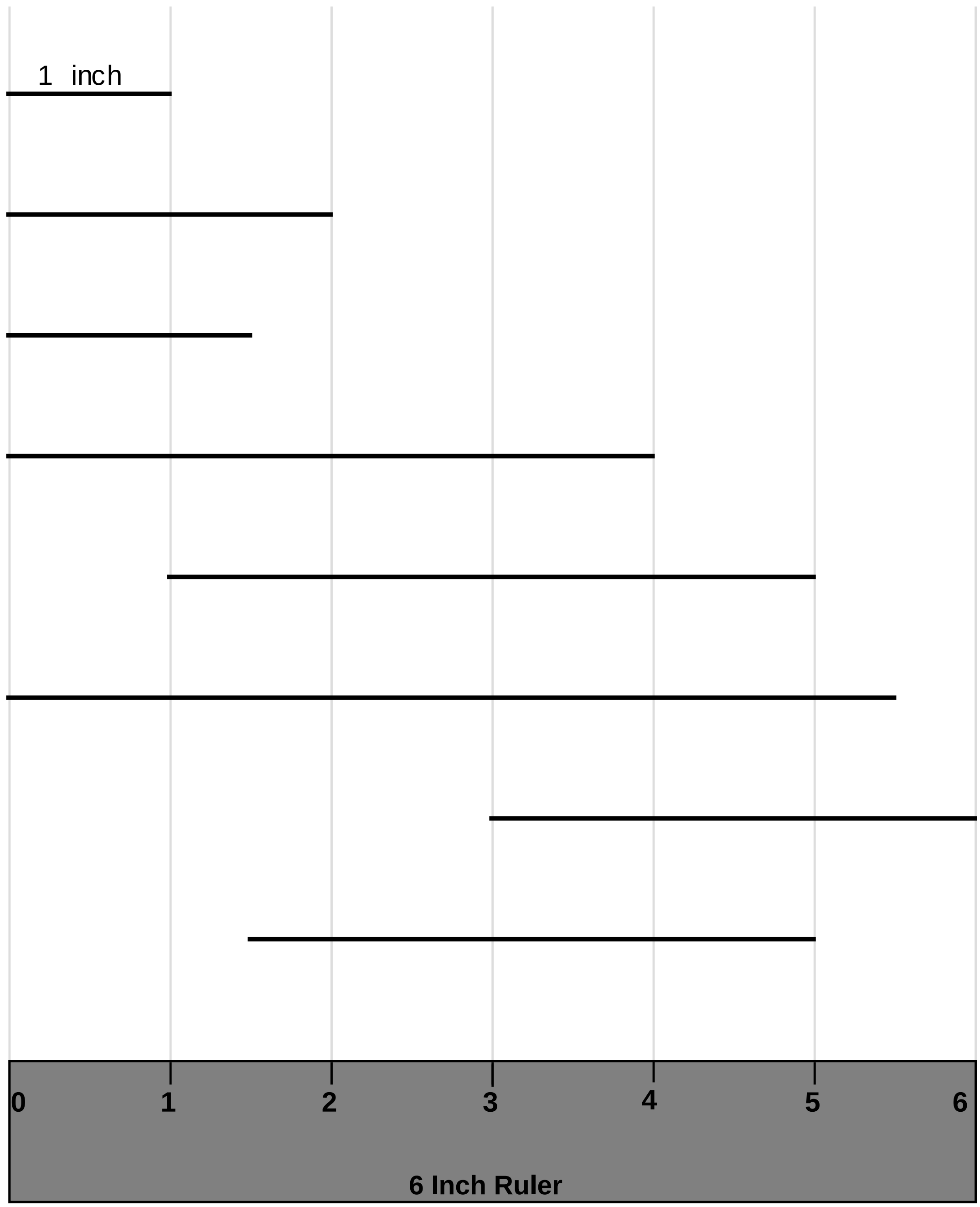
If the length of strip E is 1 unit, how long is line segment G?

If the length of strip F is 1 unit, how long is line segment G?



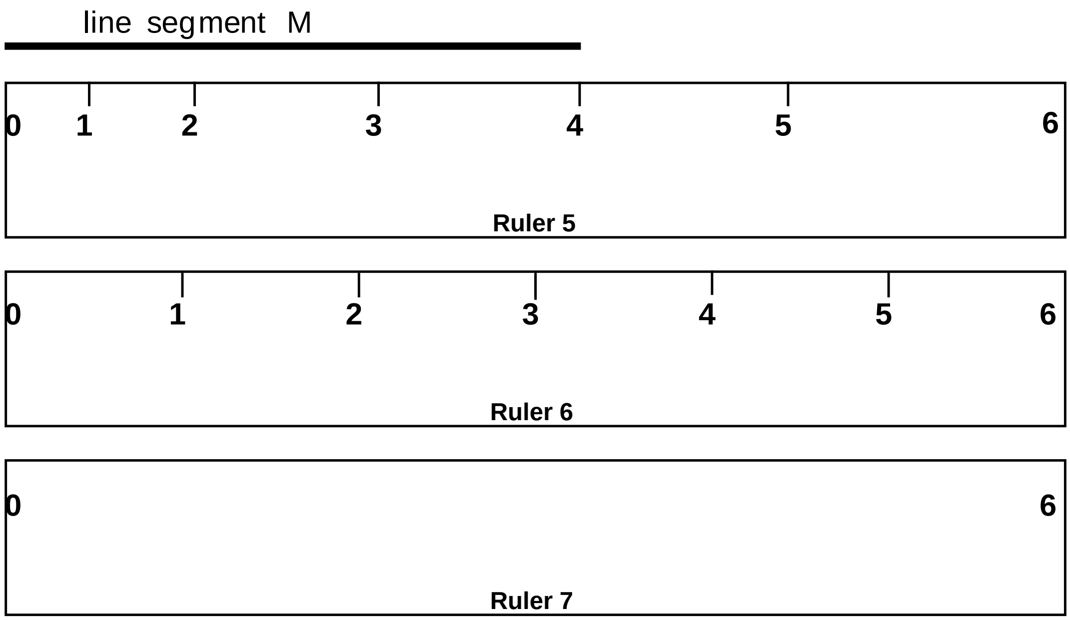
NAME: **New Student Activity Sheet C**

Above each line segment, write its length.
The first line segment is an example.



Instructions for Measuring Line Segments

Write your instructions on the lines below.



Can rulers 5, 6 and 7 be used to measure line segment M?
For each ruler, explain your answer:

Can Ruler 5 be used to measure line segment M?
Explain your answer:

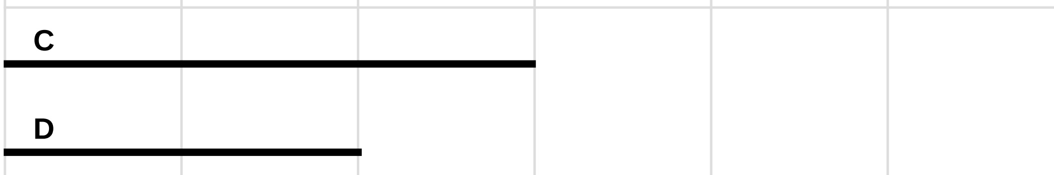
Can Ruler 6 be used to measure line segment M?
Explain your answer:

Can Ruler 7 be used to measure line segment M?
Explain your answer:

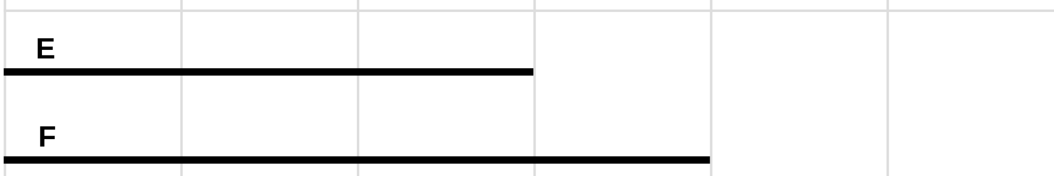
NAME: **New Student Activity Sheet F**

In this activity, you will compare the lengths of two line segments. You can use the gray grid to help you measure the line segments.

How long is line segment A, compared to line segment B?

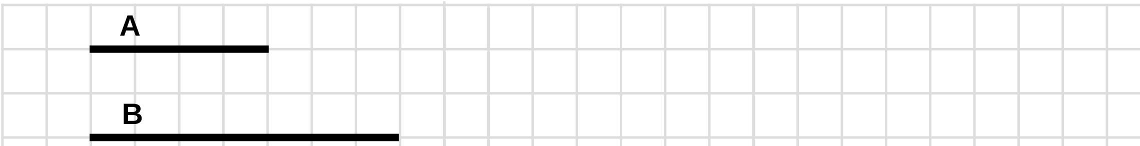


How long is line segment C, compared to line segment D? (This is a tricky one.)

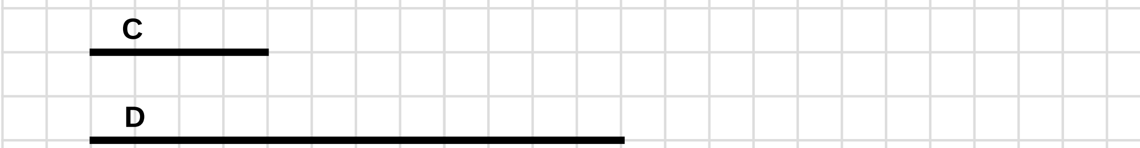


How long is line segment E, compared to line segment F?

In this activity, you will compare the lengths of two line segments. You can use the gray grid to help you measure the line segments. Hint: You don't have to use the length of 1 square as your measuring unit. Your unit could be 2 or more squares long.



How long is line segment A, compared to line segment B?



How long is line segment C, compared to line segment D?

How long is line segment E, compared to line segment F?

NAME: ____ New Student Activity Sheet J

Graph Paper

How many halves equal one whole?
How many fourths equal one whole?
How many tenths equal one whole?
How many fifteenths equal one whole?

How many fourths equal one-half?
Draw a picture to show that your answer is correct.

How many eighths equal one-half?
Draw a picture to show that your answer is correct.

How many eighths equal one-fourth?
Draw a picture to show that your answer is correct.

How many sixteenths equal one-eighth?

How many sixteenths equal one-half?

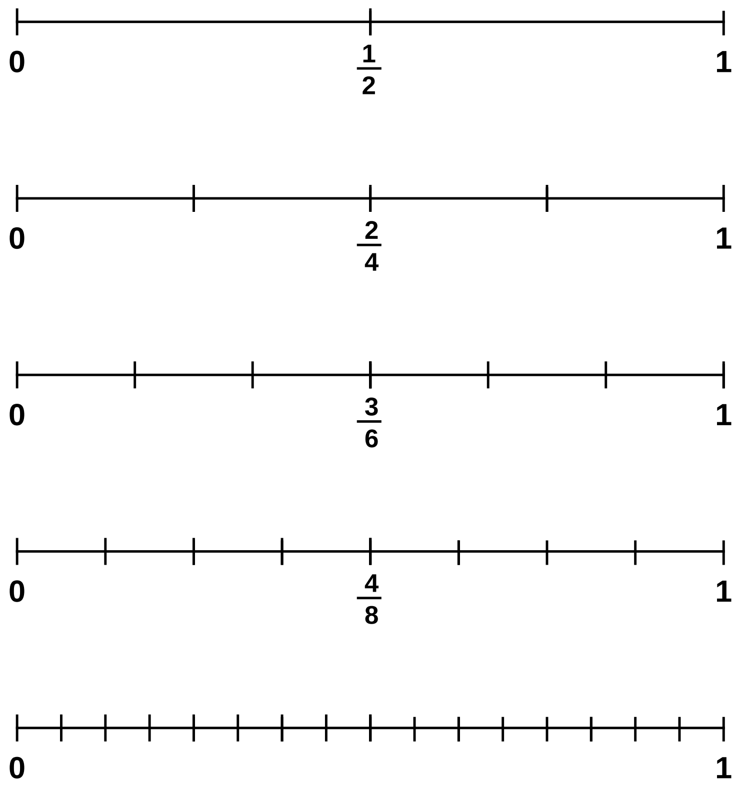
How many sixteenths equal one-fourth?

In the space below, write at least 3 fractions equal to one-fourth.

NAME: ______________________________ New Student Activity Sheet L

How many fractions are equal to $\frac{1}{2}$?

All the line segments below are the same length, 1 unit long. Each line segment has been divided into equal-sized intervals.



What do the pictures above show?

Use the pictures above to find some fractions that are equal to $\frac{1}{2}$.
Write these fractions in the space below.

Use the pictures above to find some fractions that are equal to $\frac{1}{4}$.
Write these fractions in the space below.

NAME: ____ New Student Activity Sheet M

Finding Equivalent Fractions

All fractions that are equal to one-half are called **equivalent** fractions.

How can you find fractions that are equivalent to $\frac{1}{2}$?

Write numbers above or below each fraction line to make fractions equivalent to $\frac{1}{2}$:

$\frac{\quad}{10}$ $\qquad$ $\frac{6}{\quad}$

$\frac{\quad}{16}$ $\qquad$ $\frac{10}{\quad}$

$\frac{\quad}{100}$ $\qquad$ $\frac{22}{\quad}$

$\frac{15}{\quad}$ $\qquad$ $\frac{\quad}{50}$

NAME: ______________________ New Student Activity Sheet N

Finding Equivalent Fractions - Advanced

Write numbers above or below each fraction line to make fractions equivalent to $\frac{2}{3}$:

$\frac{\_\_}{6}$ $\frac{6}{\_\_}$

$\frac{\_\_}{12}$ $\frac{10}{\_\_}$

Write numbers above or below each fraction line to make fractions equivalent to $\frac{5}{4}$:

$\frac{\_\_}{8}$ $\frac{10}{\_\_}$

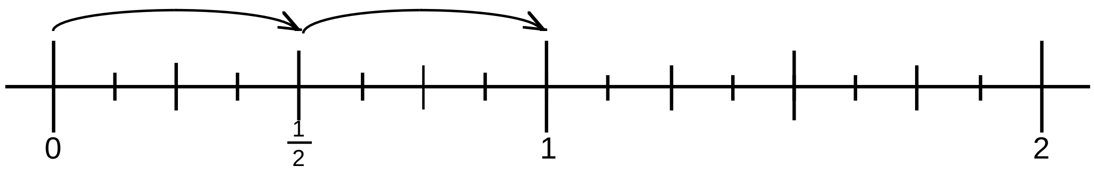
$\frac{\_\_}{16}$ $\frac{15}{\_\_}$

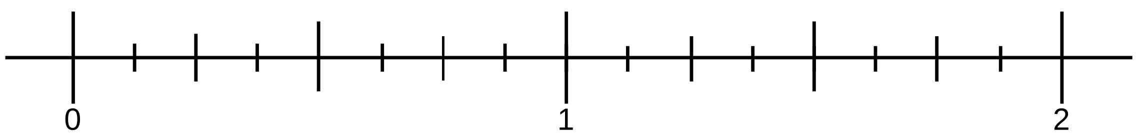
$\frac{\_\_}{100}$ $\frac{\_\_}{200}$

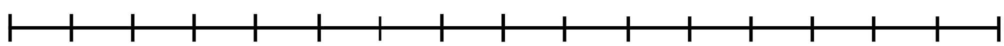
$\frac{100}{\_\_}$ $\frac{\_\_}{28}$

NAME: ______________________________ **New Student Activity Sheet P**

The diagram below shows that $\frac{1}{2} + \frac{1}{2} = 1$.







Appendix B

Sample Activity Group Lesson

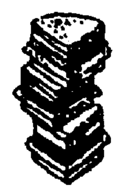
Take $\frac{1}{4}$ of 3 identical sandwiches.

Here are the 3 identical sandwiches.

Let's pile them up.

Divide the 3 sandwiches into 4 equal parts.

Take 1 part. That's $\frac{1}{4}$ of the 3 sandwiches.



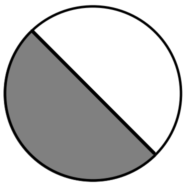
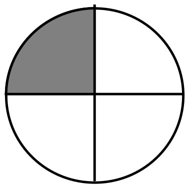
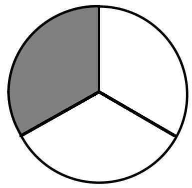
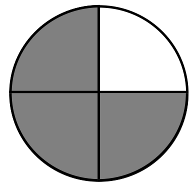
$\frac{1}{4}$ of the 3 sandwiches is the same amount as $\frac{3}{4}$ of 1 sandwich.

1. Would you have more pieces if you had $\frac{1}{4}$ of 3 sandwiches or if you had $\frac{3}{4}$ of 1 sandwich?
2. Would you have more to eat if you had $\frac{1}{4}$ of 3 sandwiches or $\frac{3}{4}$ of 1 sandwich?

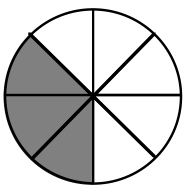
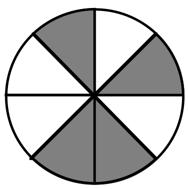
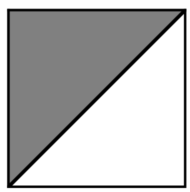
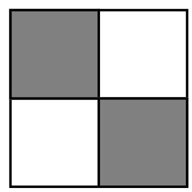
Appendix C

Pretest

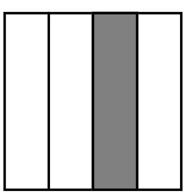
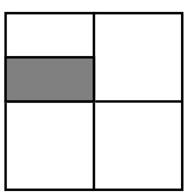
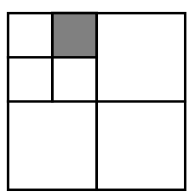
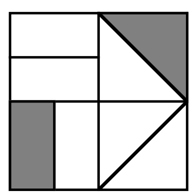
1) For each picture below, write a fraction to show what part is gray:

a.____________ b.___________ c.___________ d.___________

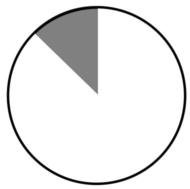
   

e.___________ f.____________ g.___________ h.___________

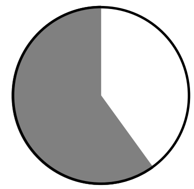
   

i.___________ j.____________ k.___________ l.___________

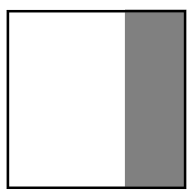
2) For each picture below, circle the fraction that shows what part of the picture is gray:



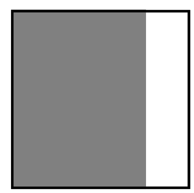
a) $\frac{1}{9}$ $\frac{1}{3}$ $\frac{3}{5}$



b) $\frac{1}{4}$ $\frac{9}{10}$ $\frac{3}{5}$

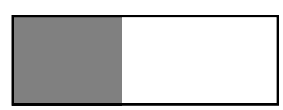


b) $\frac{1}{10}$ $\frac{1}{3}$ $\frac{2}{4}$



d) $\frac{9}{10}$ $\frac{3}{4}$ $\frac{2}{5}$

c) $\frac{11}{12}$ $\frac{2}{5}$ $\frac{5}{8}$



d) $\frac{2}{10}$ $\frac{2}{5}$ $\frac{2}{3}$

Add or subtract the fractions below:

3) $\begin{array}{r} \frac{3}{5} \\ + \frac{1}{5} \\ \hline \end{array}$

4) $\begin{array}{r} \frac{2}{10} \\ + \frac{2}{5} \\ \hline \end{array}$

5) $\begin{array}{r} \frac{1}{3} \\ + \frac{1}{2} \\ \hline \end{array}$

6) $\begin{array}{r} 7\frac{5}{8} \\ + 4\frac{1}{2} \\ \hline \end{array}$

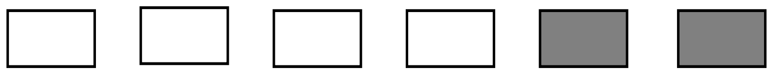
7) $\begin{array}{r} \frac{7}{10} \\ - \frac{1}{10} \\ \hline \end{array}$

8) $\begin{array}{r} \frac{5}{6} \\ - \frac{1}{3} \\ \hline \end{array}$

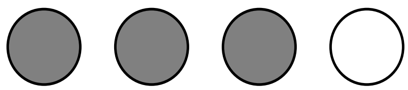
9) $\begin{array}{r} \frac{2}{3} \\ - \frac{1}{2} \\ \hline \end{array}$

10) $\begin{array}{r} 2\frac{1}{2} \\ - 2 \\ \hline \end{array}$

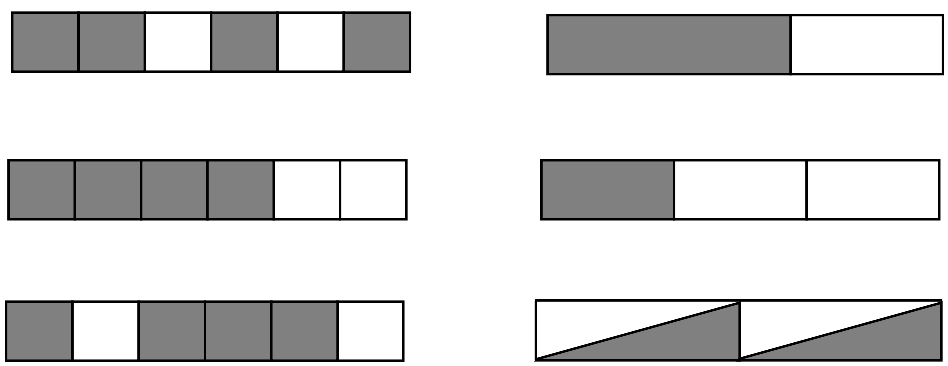
11) a. What fraction of the cards is gray? ______



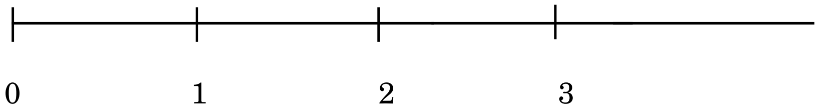
b. What fraction of the marbles is gray? ______



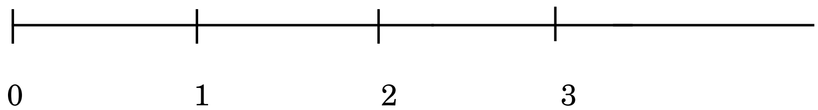
12) Circle all the pictures below that show $\frac{1}{2}$.



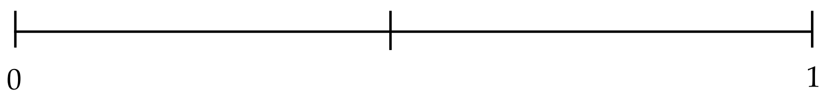
13) a. Mark with an arrow ($\downarrow$) where 4 goes on the number line below.



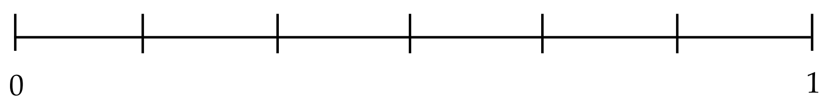
b. Mark with an arrow ($\downarrow$) where $2\frac{1}{4}$ goes on the number line below.



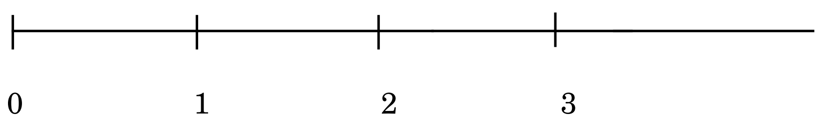
c. Mark with an arrow ($\downarrow$) where $\frac{1}{2}$ goes on the number line below.



d. Mark with an arrow ($\downarrow$) where $\frac{4}{6}$ goes on the number line below.



e. Mark with an arrow ($\downarrow$) where $\frac{2}{3}$ goes on the number line below.



14) Write one fraction that is the same as each fraction below.

Example: $\frac{1}{2} = \frac{2}{4}$

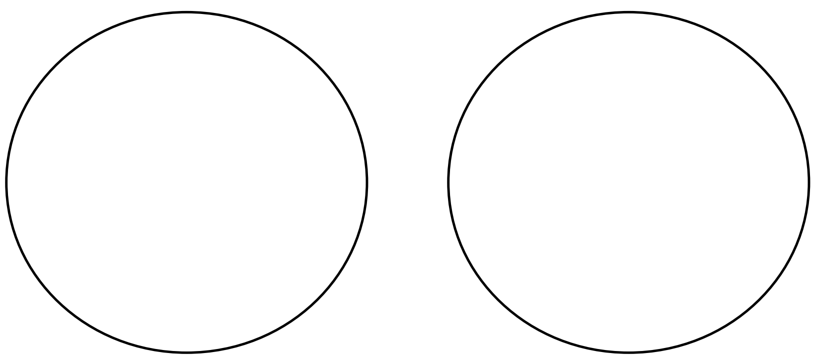
a. $\frac{2}{6} =$

b. $\frac{1}{5} =$

c. $\frac{12}{16} =$

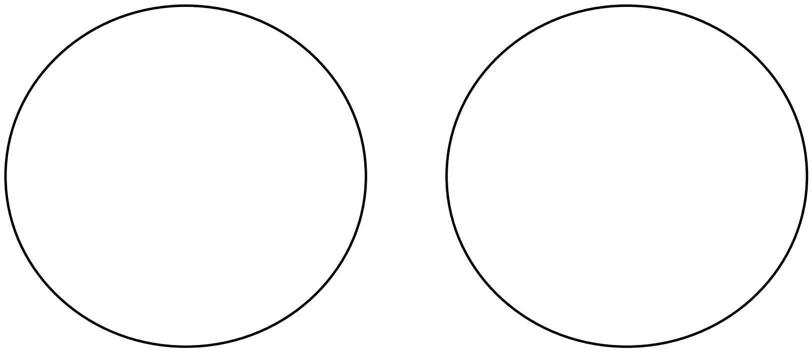
d. $\frac{7}{6} =$

15) a. Four people are going to share these two pizzas equally. Color in **one** person's part.



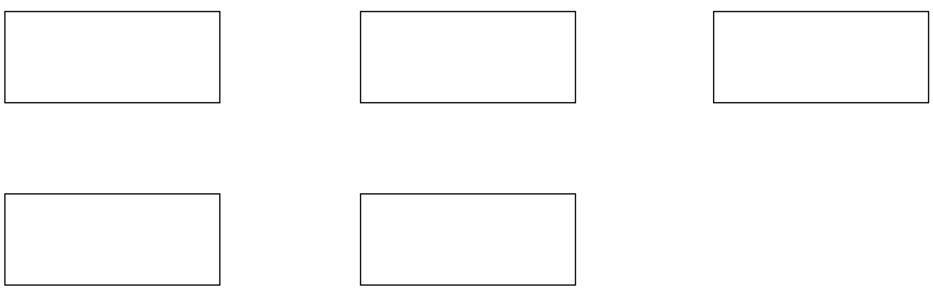
b. Write a fraction that shows how much **one** person gets ________.

16) a. Three people are going to share these pizzas equally. Color in **one** person's part.



b. Write a fraction that shows how much **one** person gets_____________.

17) a. Six people are going to share these five chocolate bars equally. Color in **one** person's part.



b. Write a fraction that shows how much **one** person gets____________.

18) Fill in the missing numbers:

a. $\frac{1}{5} = \frac{\boxed{}}{10}$

b. $\frac{3}{4} = \frac{\boxed{}}{8}$

c. $2\frac{1}{2} = \frac{\boxed{}}{2}$

d. $3\frac{1}{4} = \frac{\boxed{}}{8}$

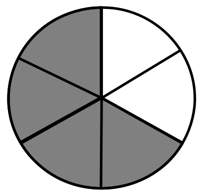
19) For each row of fractions below, show which fraction is the greatest and which fraction is the least:

a. $\frac{2}{7}$ $\frac{5}{7}$ $\frac{4}{7}$ Greatest? ___________ Least? _______

b. $\frac{1}{8}$ $\frac{1}{7}$ $\frac{1}{6}$ Greatest? ___________ Least? _______

c. $\frac{4}{7}$ $\frac{3}{8}$ $\frac{1}{2}$ Greatest? ___________ Least? _______

20) Circle a, b, c, or d below to show what part of this circle is gray:



a. $\frac{1}{2} + \frac{1}{3}$ b. $\frac{3}{6} + \frac{1}{6}$ c. $1 + \frac{1}{3}$ d. 4

21) John ran $\frac{2}{5}$ of a mile on Thursday and $\frac{3}{5}$ of a mile on Friday.
How far did he run altogether on the two days? ___________

Draw a picture to show your work:

22) a. How many fractions are equal to $\frac{1}{2}$? __________

b. Write some examples in the box:

c. Explain your answer:

Appendix D

Posttests:
Representational Knowledge
Computation
Declarative/Conceptual Knowledge
Problems and Justifications
Extended Explanation

Administration Directions

Representational Knowledge

Understanding Fractions

Name____

Age___________________

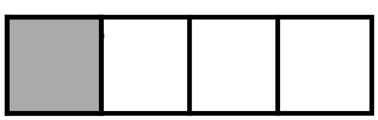
Teacher____

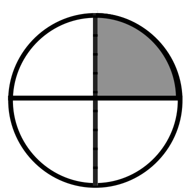
School____

Please circle one of these: Boy Girl

Below are some examples of items you will see on the next several pages. You do not have to mark anything on this page.

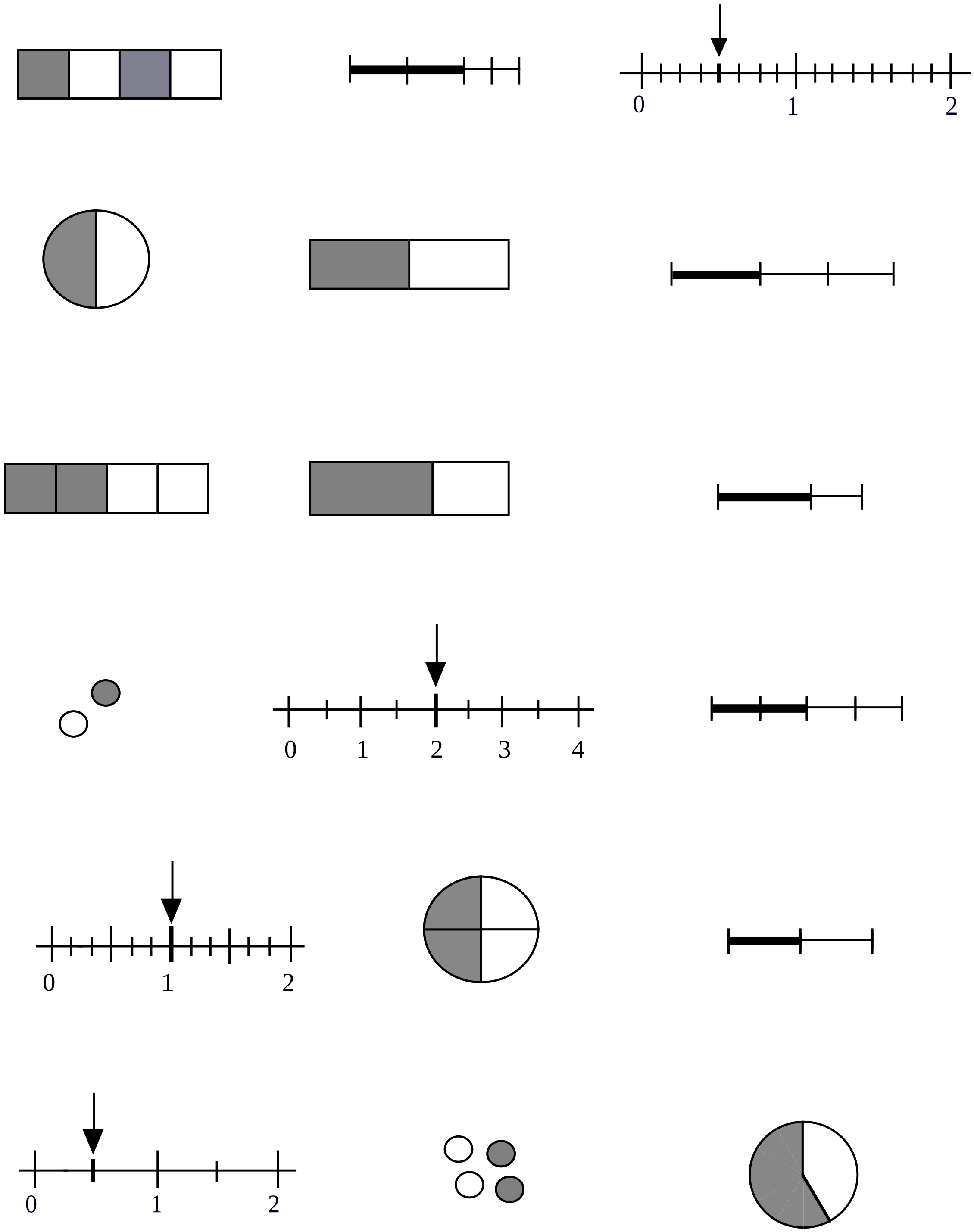
The items below show the same amount as $\frac{1}{4}$:



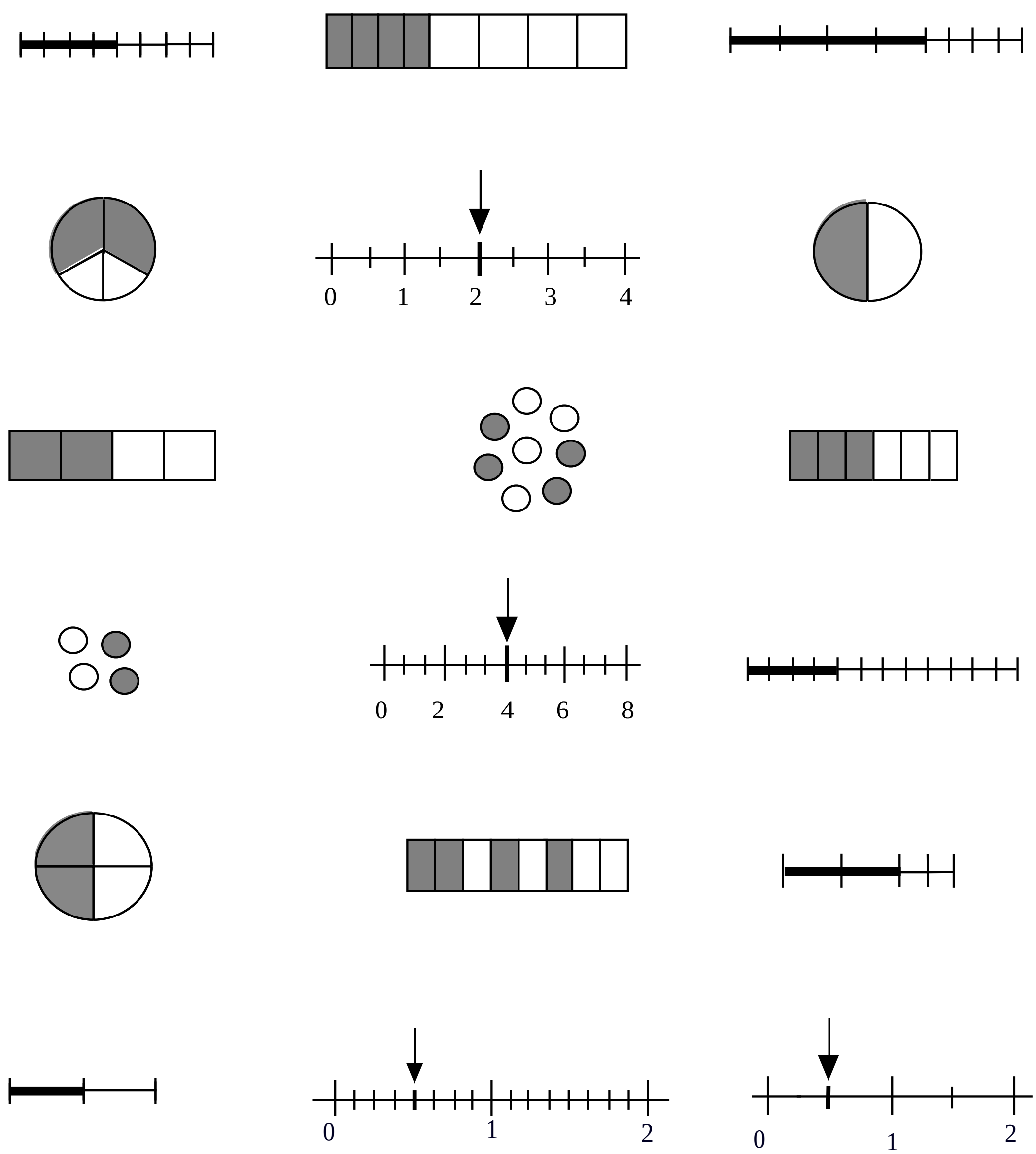


Please turn the page and follow the directions.

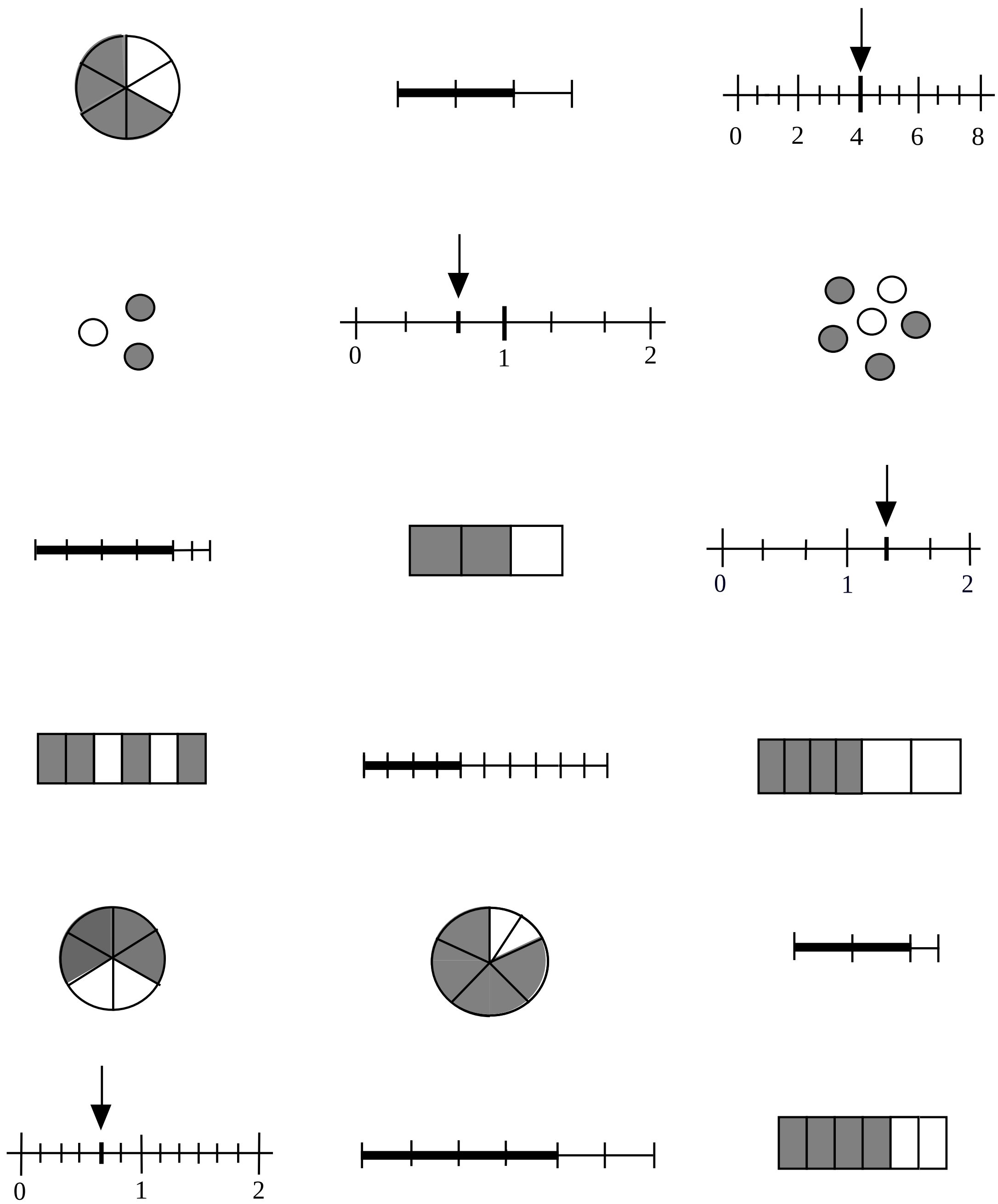
Circle the items below that show the same amount as $\frac{1}{2}$:



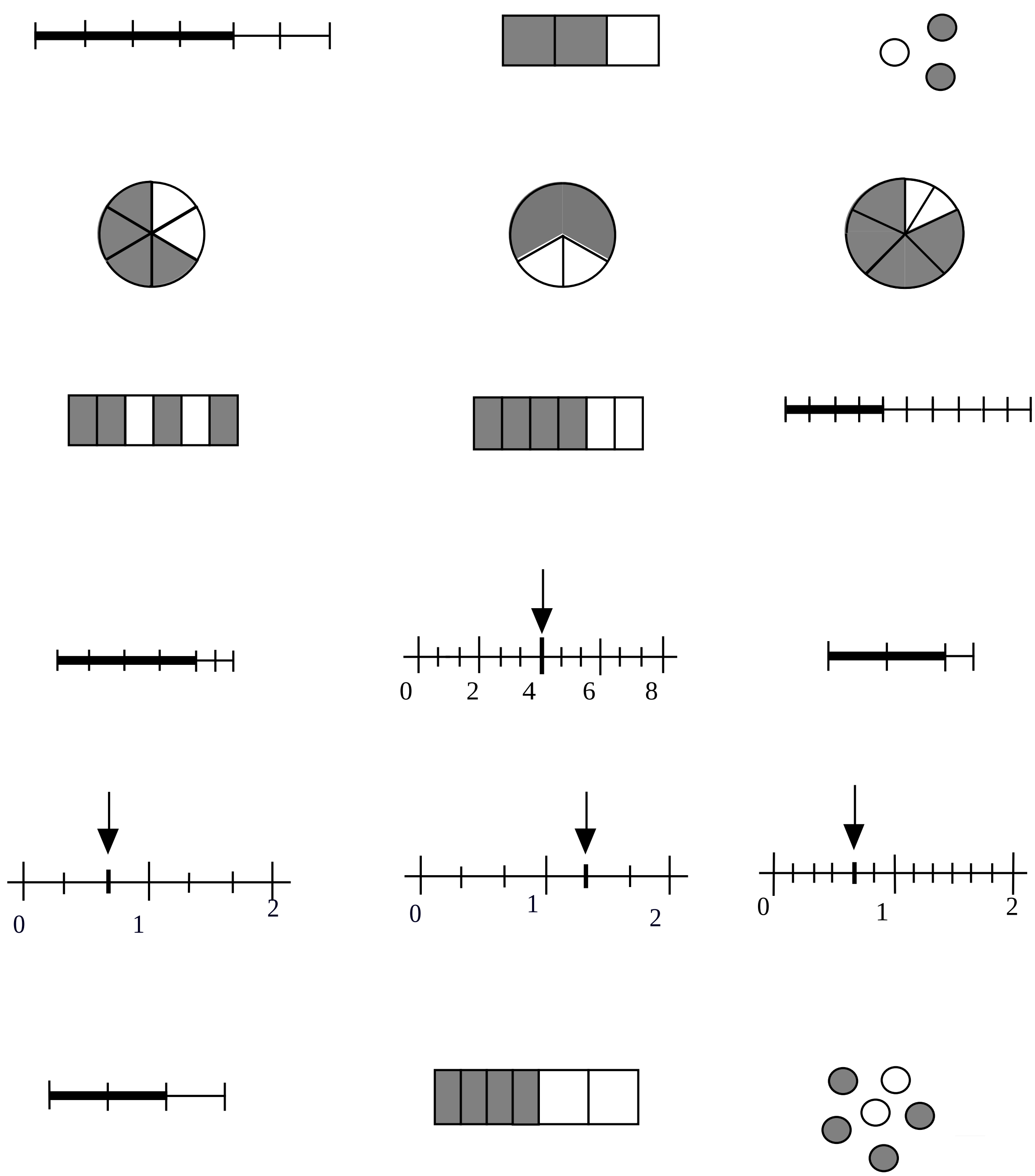
Circle the items below that show the same amount as $\frac{2}{4}$:



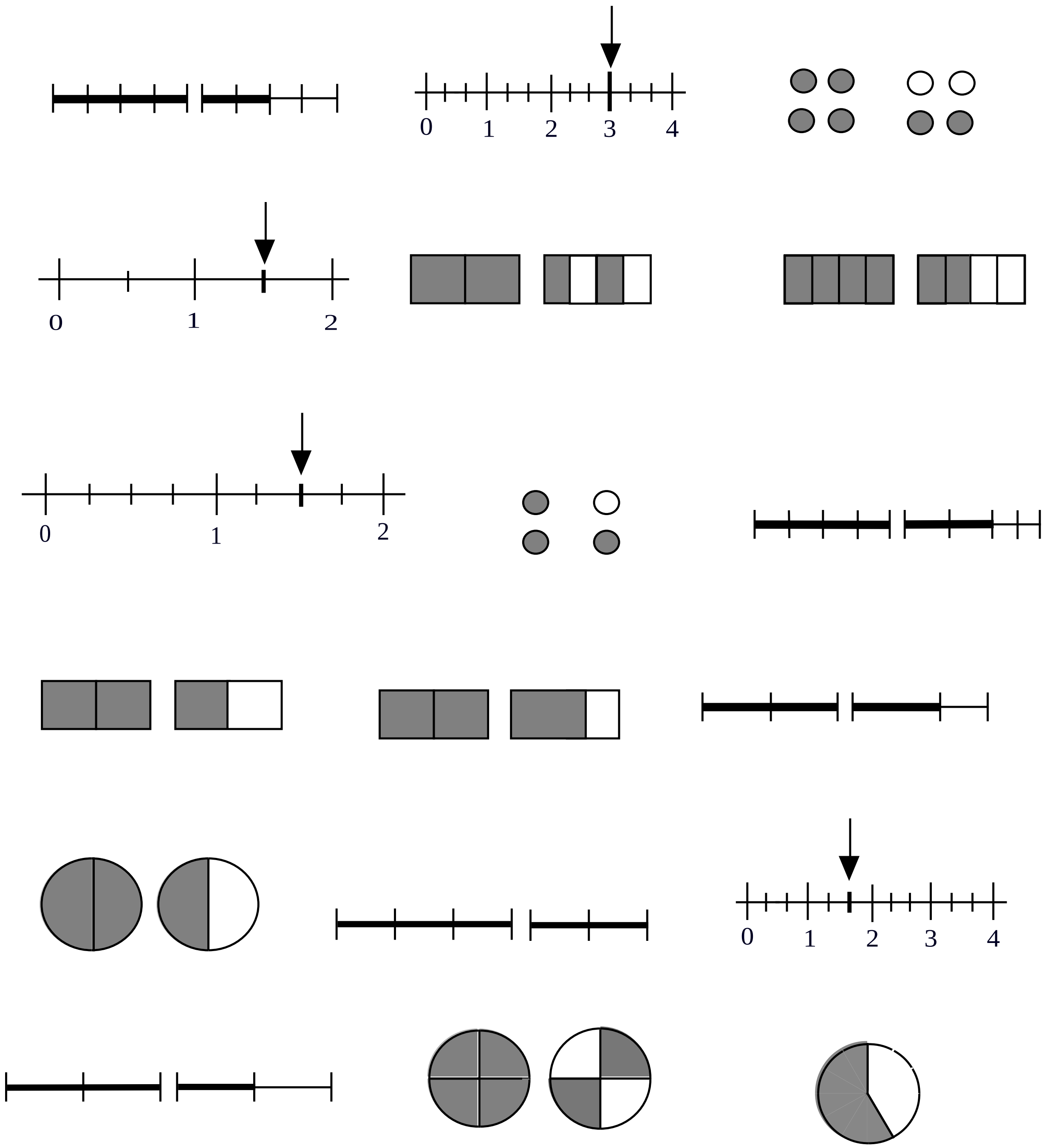
Circle the items below that show the same amount as $\frac{2}{3}$:



Circle the items below that show the same amount as $\frac{4}{6}$:



Circle the items below that show the same amount as $\frac{3}{2}$:



Computation

Add or subtract the fractions. Circle the answer:

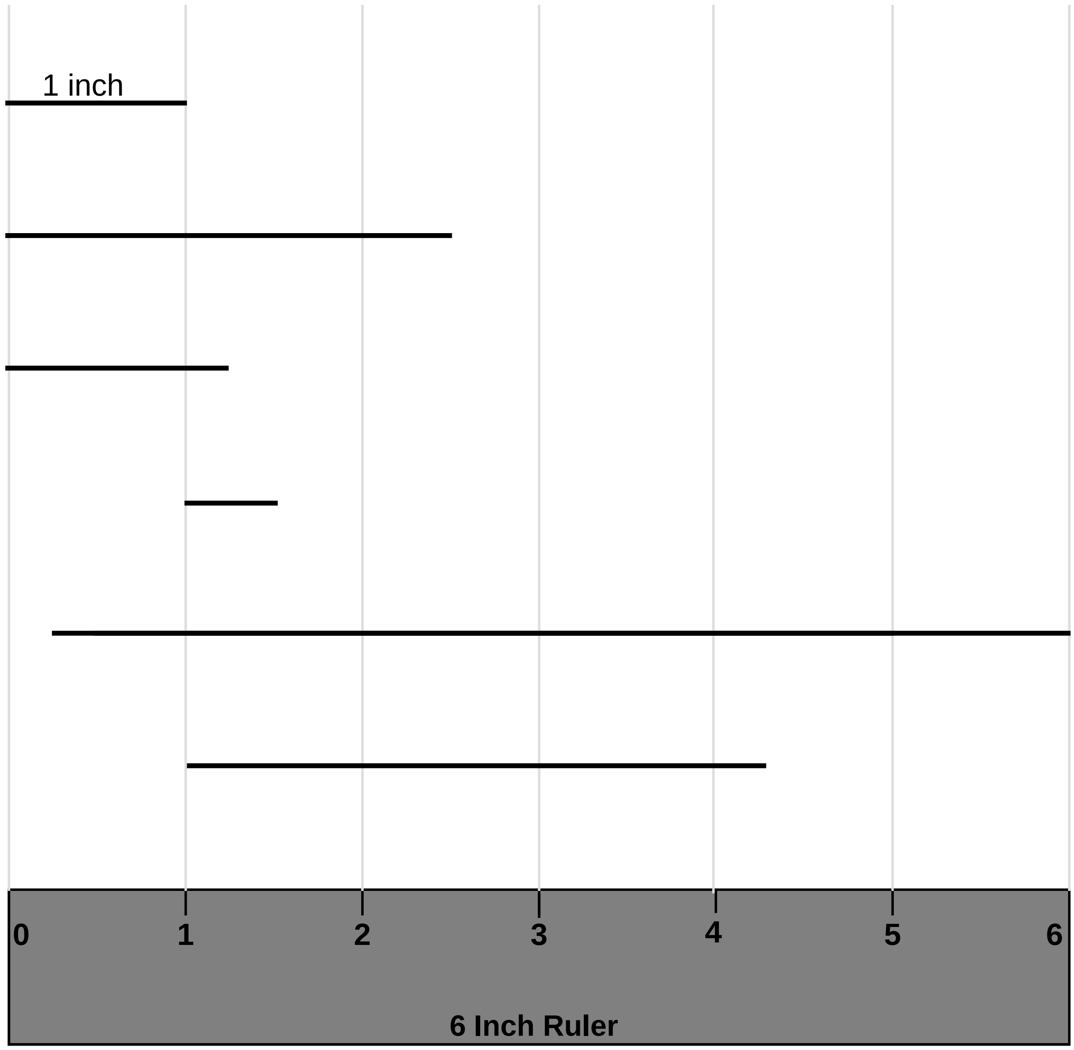
1)	$\frac{3}{7} + \frac{2}{7} =$	$\frac{5}{7}$	$\frac{5}{14}$	$\frac{7}{14}$	$\frac{14}{21}$
2)	$5\frac{1}{6} + \frac{4}{6} =$	$5\frac{5}{12}$	$5\frac{5}{6}$	$5\frac{6}{24}$	$5\frac{1}{2}$
3)	$\frac{2}{3} + \frac{1}{3} =$	$\frac{3}{6}$	3	$\frac{2}{9}$	1
4)	$\frac{5}{8} - \frac{3}{8} =$	2	$\frac{2}{16}$	$\frac{2}{8}$	$\frac{4}{5}$
5)	$\frac{10}{12} - \frac{5}{12} =$	$\frac{5}{24}$	$\frac{5}{12}$	$\frac{7}{12}$	5
6)	$8\frac{1}{3} - \frac{1}{3} =$	8	$7\frac{2}{3}$	7	$7\frac{1}{6}$

Equivalent fractions. Circle the answer:

7) $\frac{1}{2}$ equals =		$\frac{2}{3}$	$\frac{3}{6}$	$\frac{2}{5}$	$\frac{4}{9}$
8) $\frac{6}{8}$ equals =		$\frac{3}{9}$	$\frac{2}{6}$	$\frac{3}{4}$	$\frac{5}{7}$
9) $\frac{2}{3}$ equals =		$\frac{8}{12}$	$\frac{3}{4}$	$\frac{3}{9}$	$\frac{5}{6}$
10) What does N equal?	$\frac{3}{8} = \frac{6}{N}$ =	12	11	16	24
11) What does N equal?	$\frac{2}{7} = \frac{N}{21}$ =	3	16	7	6

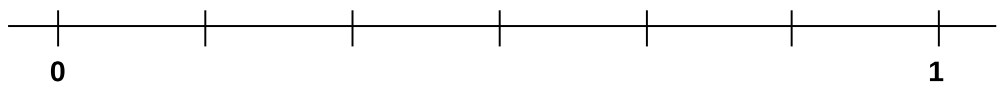
Declarative/Conceptual Knowledge

1) Above each line segment, write its length. The first line segment is an example.

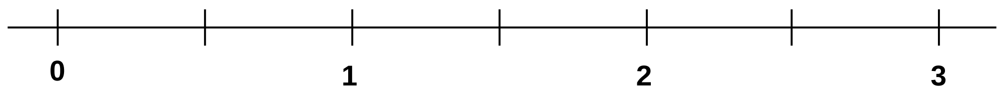


2)

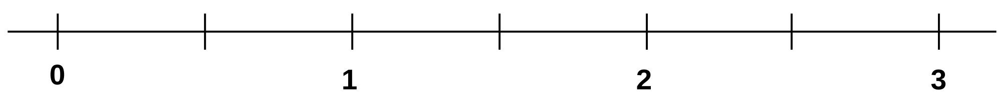
a. Mark with an arrow (↓) where $\frac{1}{3}$ goes on the number line below.



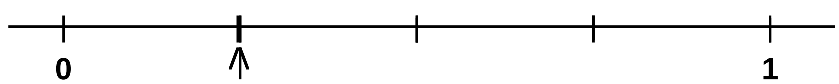
b. Mark with an arrow (↓) where $\frac{2}{3}$ goes on the number line below.

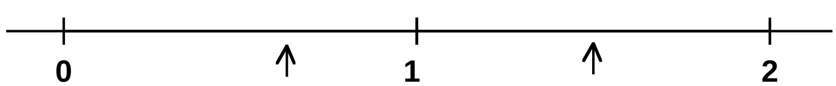


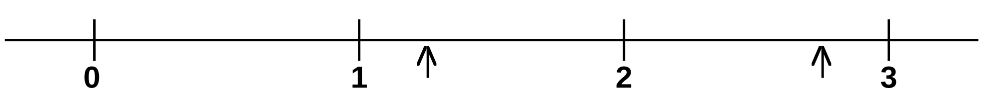
c. Mark with an arrow (↓) where $\frac{3}{2}$ goes on the number line below.



3. Write the correct numbers under each of the arrows below.
You might have to estimate.

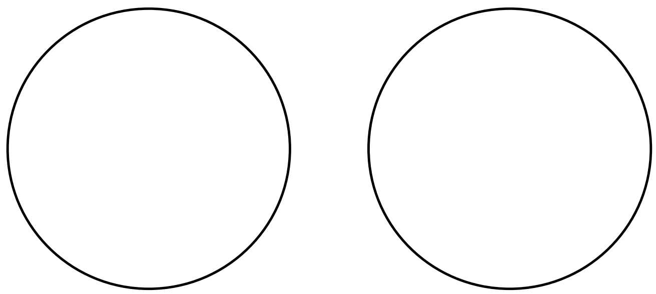






1)

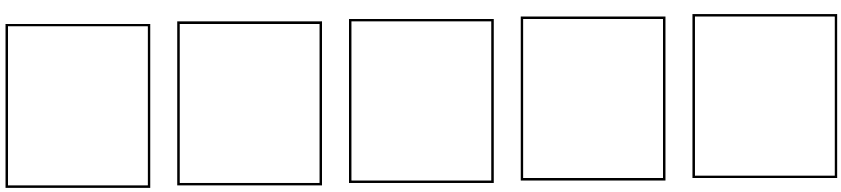
a. Four people are going to share these two pizzas equally. Color in one person's part.



b. Write a fraction that shows how much one person gets ________

2)

a. Three people are going to share these five chocolate bars equally. Color in one person's part.



b. Write a fraction that shows how much one person gets ________

3)

a. What is $\frac{1}{2}$ of 6 sandwiches?

b. What is $\frac{1}{3}$ of 12 sandwiches?

c. What is $\frac{1}{2}$ of 3 sandwiches?

d. What is $\frac{1}{3}$ of 9 dollars?

e. What is $\frac{3}{4}$ of 8 sandwiches?

f. What is $\frac{4}{3}$ of 9 dollars?

4) Three children were sharing a pizza. The cut the pizza into 12 equal slices. Maria ate 4 slices, Bob ate 3 slices and Sharon ate 6 slices.

a. What part of the pizza did Maria eat?

b. What part of the pizza did Bob eat?

c. What part of the pizza did Sharon eat?

d. What part of the pizza was left over?

Additional Declarative/Conceptual Knowledge

Problems and Justifications

1) How many ninths equal one whole?

2) How many fifths equal two wholes?

3) How many fourths equal $3\frac{1}{4}$?
Show your work.

4) How many thirds equal $2\frac{2}{3}$?
Show your work.

5) In the space below, draw at least 4 different kinds of pictures that show the fraction $\frac{3}{6}$. Then explain why the pictures show $\frac{3}{6}$.

Some problems on the next few pages ask you to explain how to solve the problem. Or you might be asked to explain a math idea. When you are asked to explain, be sure to put down everything you know. Imagine that you are explaining to someone who doesn't know anything about math. Your explanations should be very clear and complete.

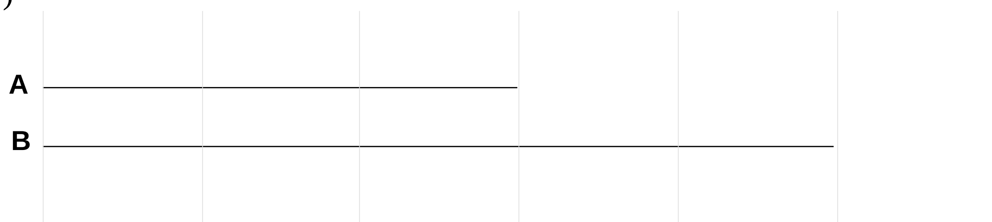
6) Which is larger $\frac{2}{5}$ of a pizza or $\frac{2}{4}$ of a pizza?
Draw a picture that shows your answer is correct.
Explain your picture.

7) Ana sliced a pie into 5 equal pieces. She ate two pieces.
How much of the pie did she eat?

8) Carlos and Lee Ann sliced a pizza into 8 equal pieces.
Carlos ate 3 pieces. Lee Ann ate 2 pieces. How much did
Carlos eat, compared to Lee Ann?

9) Which is greater, $\frac{3}{5}$ of a mile or $\frac{1}{2}$ of a mile?
Draw a picture that shows your answer is correct.
Explain your picture.

10)



How long is line segment A, compared to line segment B?

11)

How long is line segment A, compared to line segment B?

12) Lug is 4 feet tall. Mab is 5 feet tall. How tall is Lug, compared to Mab?

13) Bek is 6 feet tall. Ruf is 4 feet tall. How tall is Bek, compared to Ruf?

14) Find a fraction between $\frac{1}{2}$ and $\frac{3}{4}$

Explain how you find the answer to this problem.
Draw a picture that shows your answer is correct.

15) Find a fraction between $2\frac{1}{2}$ and $2\frac{3}{4}$

Explain how you find the answer to this problem.
Draw a picture that shows your answer is correct.

16) Draw a picture to show whether this is true or not:

$$\frac{1}{2} + \frac{1}{6} = \frac{4}{6}$$

Explain how your picture shows the answer.

17) Draw a picture to show whether this is true or not:

$$\frac{3}{4} = \frac{6}{8}$$

Explain how your picture shows the answer.

18) Draw a picture to show whether this is true or not:

$$\frac{4}{3} = 1\frac{2}{6}$$

Explain how your picture shows the answer.

19) Draw a picture to show which of these two fractions is larger:

$$\frac{2}{3} \text{ or } \frac{3}{4}$$

Explain how your picture shows the answer is correct.

STOP

Wait for your teacher to tell you
when to go on to the next page.

Extended Explanation

Explanation Task

Imagine a person from a television station has asked you to give a demonstration on TV. You will be on a show to help other students learn about math. You are asked to explain everything fifth grade students should know about fractions.

Below are some questions you should try to answer. These are questions that students in the TV audience will ask you.

For each question you should draw as many pictures as you can to show what you mean. Then write down what you would say about your pictures on TV. Use as many words and pictures as you need.

What is a fraction?

Why are there two numbers in a fraction?

How many fractions are there between 0 and 1?

How many fractions are equal to 1/2?

What other important ideas should students know about fractions? Show how you would explain these ideas. Use as many pictures and words as you need. If you need more space,
continue on to the next page.

Administration Directions
and
Teacher Log

INSTRUCTIONS
FOR FIRST DAY OF
FINAL ASSESSMENTS

This package contains materials for the first day of testing.

GENERAL NOTES

Estimated time for the enclosed assessments is one 40 minute period. If your class periods are shorter than this, testing may have to be extended to the next day. If any students are absent on a testing day, please have them complete any tests they miss when they return.
Instructions in boldface type are to be read aloud to the students. Directions for administering the test are in plain type.
Students will complete three sets of papers on the first day. If all students finish working on a given set before the allotted time has passed, you may introduce the next set. If there is not enough time in the period left to complete a new set of papers, save that set for your next testing period. Just make sure to administer the sets in the prescribed sequence, even if it takes more than two days to do it.

Distributing the papers:

In this package there are three separate piles: 1) "Understanding Fractions," 2) "Add or subtract fractions," and 3) Conceptual Items. Within some piles, test forms are not identical. For technical reasons, it is extremely important that you distribute papers according to the following plan. **The papers need to be distributed in the same order as your class roster.** The easiest way to do this (without having to do any prior work) is to set out the three different piles, then call out names in order from your roster. Each student should come up and take one stapled set (the top one) from each pile. Use this plan at the point shown below.

Begin: **"For the next two days you'll be working on fraction tasks. Some of the tasks will be like the ones you've been doing this week and last week, and some of the tasks are different. These items will show what you know about fractions. There are a lot of different items, so you'll have a lot of chances to show what you know. Don't worry if you can't do all the items. Just skip the ones you can't do or try to guess what the answer might be. Some of the tasks may be easy for you and some unfamiliar. But it is important that you try your best to show what you know. I'll be looking at your papers, and some people at UCLA who are studying fractions, will look at your papers also."**

Before distributing the tests, say, **"Don't start working on the items until everyone has their papers. I will tell you when to start."**

DISTRIBUTE PAPERS ACCORDING TO ABOVE PLAN.

After all tests have been distributed, say:

"The first thing you should do is print your name on the first page of each set of papers."

Check to see that this happens.

"Now take the set that says 'Understanding Fractions' at the top; put the other sets aside. On the page that says 'Understanding Fractions,' fill in your age, my name, and the name of this school, on the lines. Then circle whether you are a boy or girl."

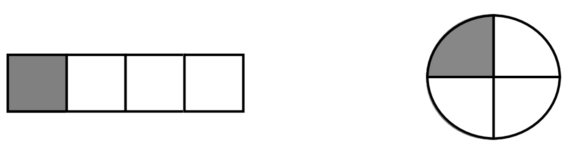
When everyone has done this, continue:

"Now I'll read what it says at the bottom. Follow along with me."

Read these lines:

Below are some examples of items you will see on the next several pages. You do not have to mark anything on this page.

The items below show the same amount as $\frac{1}{4}$:



Please turn the page and follow the directions.

Say: **"Now turn to the next page. On this page, look at each item and circle the ones that show the same amount as the fraction at the top. Then go on to the next page. Does everyone understand?"**

"Now you may begin to work on the items. Be sure to make your circles dark enough to see. You have about 20 minutes to complete the sheets. If you need help, raise your hand to ask for help."

If students ask for help, it is permissible to assist them in reading and understanding the directions. For example, you can point out where the answers should be written. It is not acceptable to discuss the mathematical content of the items. You should not say, for instance: **"This picture shows 4 parts shaded out of 8. What fraction is that?"**

After 15 minutes, say:

"You have five more minutes to finish the task."

After 5 more minutes:

"Time is up. Please stop now and turn in your sheets."

Collect these papers.

"Next we'll do the set that says 'Add or subtract the fractions' near the top. Make sure your name is on this page. Read the directions to yourself and make sure you know what to do. If you know what to do, begin working. If you have a question, raise your hand. When you're done with this page, continue onto the next page. You have 8 minutes."

After 6 minutes, say:

"You have two more minutes to finish the task."

After 2 more minutes:

"Time is up. Please stop now and turn in your sheets."

Collect these papers.

"Next we'll do the final set of papers. Your set might not look exactly like everyone else's. Make sure your name is on the top of the first page. Read the directions to yourself and make sure you know what to do. If you know what to do, begin working. If you have a question, raise your hand. Keep working until you've finished all the pages. If you have time left over, you can check your answers. You have 12 minutes for this part."

After 9 minutes, say:

"You have three more minutes to finish the task."

After 3 more minutes:

"Time is up. Please stop now and turn in your sheets."

Collect these papers.

This is the end of the first day's assessment.
Keep all papers and return them to us in a Federal Express box with the second day's papers. (This box will be sent with the second-day assessments.)
Thank you for your help.

INSTRUCTIONS
FOR
FINAL ASSESSMENTS

This package contains materials for the final day of testing, plus a short Teacher Log we need to have you fill out. It may be possible to complete this log during the first 25-minute testing period (described below). We are adding some additional paid time for everyone in case you cannot complete this form during the school day.

GENERAL NOTES

Estimated time for the enclosed assessments is one 40-45 minute period. If your class period is shorter than this, testing may have to extend to the next day. If any students are absent, please have them complete any tests they miss when they return.
Instructions in boldface type should be read aloud to the students.
Administrative directions are in plain type.
Two main types of assessments will be given during this period: 1) problem solving and short explanation tasks, and 2) an extended explanation task. It will take about 25 minutes to complete the first group of tasks, and 15 minutes to do the extended explanation. All tasks have been assembled into one package.

Begin: **"Today we'll continue with the fraction assessments we started yesterday. There are a lot of different items, so you'll have a lot of chances to show what you know about fractions. Don't worry if you can't do all the items. Some of these items are hard even for adults. Just skip the ones you can't do or try to guess what the answer might be. Remember, it is important that you try your best to show what you know. I'll look at your papers, and some people at UCLA who are studying fractions, will look at them also."**

Before distributing the tests, say, **"Don't start working on the items until everyone has a package. I will tell you when to start."**

Distribute papers as follows:

Each student will receive one assessment package. **The packages must be distributed in the same order as your class roster.** The top paper in the pile should go to the first student on your roster, and so on.

After all tests have been distributed, say:

"The first thing you should do is print your name and the date on the first page. Write your first and last name."

Check to see that this happens.

"Wait until I tell everyone to begin. After you begin, you may continue working until you see the word 'Stop'" At that point you can go back to any problems you skipped, or you can check your answers. When you are asked to explain something, imagine you are writing for someone who doesn't know anything. Write as much as you can, even if it seems obvious. Are there any questions?"

"Now you may begin to work on the problems. You have about 25 minutes to complete the sheets. If you have a question, raise your hand to ask for help."

If students ask for help, it is permissible to assist them in reading and understanding the directions. For example, you can point out where the answers should be written. It is not acceptable to discuss the mathematical content of the items.

After 20 minutes, say:

"You have five more minutes to finish the task."

(Note: If all students complete the first set of problems and stop working before the allotted time has passed, you may take a short break and go on to the next task.)

After 5 more minutes:

"Time is up. Please stop now and close up your package."

It is highly advisable that you take a short break at this time (at least a few minutes).

The final task will take 15 minutes. If there is not enough time remaining in the period to complete this task, postpone it until the next day.

<u>Administering the Extended Explanation Task</u>

"Turn to page 7 of your assessment package. Page numbers are in the bottom left corner. The page you want looks like this:"

Hold up page 7. (A copy is attached; the heading is "Explanation Task.")

"Now I'll read the directions. Follow along with me."

Read the three short paragraphs at the top of the page.

"You may begin writing. Remember to be as clear and complete as you can."

After 12 minutes, say:

"You have three more minutes to finish the task."

After 3 more minutes:

"Time is up. Please stop now and turn in your papers."

Collect all papers.

"We've finished this fractions project but you'll be learning more about them later. Fractions are very important in mathematics and everything you have learned about them will help you in school and outside of school. You'll see fractions many times again in the future."

This is the end of the final assessments.

Teacher Log

Name:____

Please give a brief description of activities during each day of the instructional and assessment periods. "Principles" teachers may list lessons or activities completed; "Activity" teachers may provide page numbers from the text or brief descriptions of activities. Don't worry if you can't remember everything with 100% accuracy--we just want to see whether there were any big discrepancies between teachers. Do not include days when nothing happened--e.g., you were out sick, there was an assembly, etc.

Day	Date	Instructional Activities
1		
2		
3		
4		
5		
6		
7		
8		Assessment Activities
9		
10		
11		

Appendix E

Classroom Observation Form

ACOT CLASSROOM OBSERVATION INSTRUMENT

Teacher Lee

A1 Site	Teacher	A2 Obs	A3 Date		
16		PC	11	12	93

PLANNED Instructional Time Available		ACTUAL Instructional Time Used	
A4 Start Time	A5 Finish Time	A6 Start Time	A7 Finish Time
11 00 AM	11 55	11 05 AM	11 30 AM

SUBJECT

Elementary

- Reading
- Math
- Writing
- Language Arts
- Science
- Social Studies
- Music
- Art
- Computer Skills
- Problem Solving

TASK CONTEXT

Number of Days to Completion
a 1 · b 2-3 · c 4-5 · d 6 or more · e

Number of Component Activities
1 · 2-3 · 4 or more

Student Initiative and Choice
none · a little · some · fair amt. · great deal

Student Initiative and Choice in:
when · content · resources · peers

NUMBER OF STUDENTS

B1 → # Assigned 26

B2
- a Individual
- b Pair/2
- c Small Group/3-8
- d Large Group/9-15
- e Almost Whole Group/16-2x
- f Whole Class

b f

Free Choice Activity

OTHER ADULTS - CODE ALL

D1
- a Co-teacher
- b Specialist
- c Student Teacher
- d Aide
- e Volunteer
- f Parent
- g Other

D2 **Total Number of Adults Present**
0 · 1 · 2 · 3 · 4 · 5 · 6 or more

3992

DO NOT MARK IN THIS AREA

Scanner Forms 360-2516-54321

ACTIVITY CONTEXT	NATURE OF RESPONSE

Ability Groupings

E1 Homogeneous	1	2	3	4	5 or more
E2 Heterogeneous					

Ability Level of Target Students (optional)

E3 Low Medium High

How Working F1 F2 F3 F4 F5 F6 F7 F8 F9

a Teacher-led	1 2 3	4 5 6	7 8 9
b Independent	1 2 3	4 5 6	7 8 9
c Small Group/Coop	1 2 3	4 5 6	7 8 9
d Small Group/Collab	1 2 3	4 5 6	7 8 9
e Pair/Tutor	1 2 3	4 5 6	7 8 9
f Pair/Coop	1 2 3	4 5 6	7 8 9
g Pair/Collab	1 2 3	4 5 6	7 8 9
h Student Choice	1 2 3	4 5 6	7 8 9
i Student-Led	1 2 3	4 5 6	7 8 9
j With Aide	1 2 3	4 5 6	7 8 9
k Competitive	1 2 3	4 5 6	7 8 9
l Comp and Coop	1 2 3	4 5 6	7 8 9

Product - Level of Processing G1 G2 G3 G4 G5 G6 G7 G8 G9

a Low/Rote	1 2 3	4 5 6	7 8 9
b Medium	1 2 3	4 5 6	7 8 9
c High/Elaborate	1 2 3	4 5 6	7 8 9
d Other (below)	1 2 3	4 5 6	7 8 9
e Writing	1 2 3	4 5 6	7 8 9
f Discussion	1 2 3	4 5 6	7 8 9
g Worksheet/Workbook	1 2 3	4 5 6	7 8 9

Product Length - Items H1 H2 H3 H4 H5 H6 H7 H8 H9

a Repeat/Copy	1 2 3	4 5 6	7 8 9
b Select	1 2 3	4 5 6	7 8 9
c Construct- SHORT	1 2 3	4 5 6	7 8 9
d Construct- MEDIUM	1 2 3	4 5 6	7 8 9
e Construct- LONG	1 2 3	4 5 6	7 8 9
f other			
g Listen	1 2 3	4 5 6	7 8 9
h Watch	1 2 3	4 5 6	7 8 9
i Read	1 2 3	4 5 6	7 8 9
j Take Notes	1 2 3	4 5 6	7 8 9

ADULT ROLE

Teacher I1 I2 I3 I4 I5 I6 I7 I8 I9

a Explain/Provide Info	1 2 3	4 5 6	7 8 9
b Question	1 2 3	4 5 6	7 8 9
c Answer Student Q's	1 2 3	4 5 6	7 8 9
d Goal Relevance	1 2 3	4 5 6	7 8 9
e Direct ongoing wk	1 2 3	4 5 6	7 8 9
f Correct/Grade	1 2 3	4 5 6	7 8 9
g Test	1 2 3	4 5 6	7 8 9
h Reflective Dialogue	1 2 3	4 5 6	7 8 9
i Facilitate Discuss	1 2 3	4 5 6	7 8 9
Feedback/Monitor/Help	1 2 3	4 5 6	7 8 9
k Confer/Diagnose	1 2 3	4 5 6	7 8 9
l Joint Problem - Solve	1 2 3	4 5 6	7 8 9
m Read to students	1 2 3	4 5 6	7 8 9
n Manage/Oversee	1 2 3	4 5 6	7 8 9
o Control/Discipline	1 2 3	4 5 6	7 8 9
p Non-English	1 2 3	4 5 6	7 8 9
q Not Present	1 2 3	4 5 6	7 8 9
r Review	1 2 3	4 5 6	7 8 9

STUDENT BEHAVIORS

Students' Focus and Investment L1 L2 L3 L4 L5 L6 L7 L8 L9

a Very high	1 2 3	4 5 6	7 8 9
b High	1 2 3	4 5 6	7 8 9
c Some	1 2 3	4 5 6	7 8 9
d Low	1 2 3	4 5 6	7 8 9
e Very low	1 2 3	4 5 6	7 8 9

Appropriateness of Student Behavior M1 M2 M3 M4 M5 M6 M7 M8 M9

a Almost all app	1 2 3	4 5 6	7 8 9
b 75% are	1 2 3	4 5 6	7 8 9
c 50% are	1 2 3	4 5 6	7 8 9
d 25% are	1 2 3	4 5 6	7 8 9
e Almost none app	1 2 3	4 5 6	7 8 9

Difficulty with the Task N1 N2 N3 N4 N5 N6 N7 N8 N9

a Almost all are	1 2 3	4 5 6	7 8 9
b 75% are	1 2 3	4 5 6	7 8 9
c 50% are	1 2 3	4 5 6	7 8 9
d 25% are	1 2 3	4 5 6	7 8 9
e Almost none are	1 2 3	4 5 6	7 8 9
f With:	Task	Tech	Both

RESOURCES IN USE

Textual	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9
a Textbook	1	2	3	4	5	6	7	8	9
b Assigned Literature	1	2	3	4	5	6	7	8	9
c Library Books	1	2	3	4	5	6	7	8	9
d Reference Books	1	2	3	4	5	6	7	8	9
e Newspapers	1	2	3	4	5	6	7	8	9
f Class Visual	1	2	3	4	5	6	7	8	9
g Wkbook/Wksheet	1	2	3	4	5	●	7	8	9
h Test	1	2	3	4	5	6	7	8	9
Reference/Help Sheet	1	2	3	4	5	6	7	8	9
j Students' Work	1	2	3	4	5	6	7	8	9
k Paper (plain)	1	2	3	4	5	●	7	8	9
l File Cards	1	2	3	4	5	6	7	8	9
m Board	1	2	3	4	5	6	7	8	9
n **Teacher prepared**	1	2	3	4	5	6	7	8	9
o None	●	●	●	●	●	6	7	8	9

Hands-On Materials	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9
a Math manipulatives	1	2	●	●	●	6	7	8	9
b Measuring tools	1	2	3	4	5	6	7	8	9
c Science materials	1	2	3	4	5	6	7	8	9
d Social sci materials	1	2	3	4	5	6	7	8	9
e Art materials	1	2	3	4	5	6	7	8	9
f Music materials	1	2	3	4	5	6	7	8	9
g Strategy games	1	2	3	4	5	6	7	8	9
h Other objects	1	2	3	4	5	6	7	8	9
i **Teacher prepared**	1	2	●	●	●	6	7	8	9
None	●	●	3	4	5	6	7	8	9

Example of observer notes.

NOTES

11:05 Used line segments on overhead
did thumbs up + down for fractional
parts of line segments
11: 15 Each child has 5 strips of
bright yellow paper – folding or
cutting to get fractional parts.

11:26 handed out graph paper
used as number lines
finding fractional parts

Appendix F

Explanation Scoring Rubric
Anchor Papers
Examples of Principles, Facts, and Misconceptions

Mathematics Explanation Scoring Rubric

1. General Impression of Content Quality (GICQ)
How much does the student know about this mathematical topic?
(1-5 point global rating: 1 = no knowledge, 5 = highest level of understanding)

2. Number of Principles or Concepts (PN)
Record number of principles or concepts that the student uses with comprehension. Principles and concepts are general, abstract ideas. To be counted, the idea must be clearly and explicitly stated; you should not have to guess whether the student knows what s/he is talking about.

Examples related to fractions include the following ideas:

- Between any two numbers, you can find an infinite number of fractions.
- For any fraction you can find an infinite number of equivalent fractions.
- When you partition or measure some object in order to find a fraction of it, the partitions or measurement units must be equal in size.
- You can partition any quantity into as many equal-sized parts as you want; to get more parts, you just have to make them smaller.
- A fraction is a number that shows a relation between two other numbers.
- See attached examples.

Note: The definition of a fraction as a part of a whole is conceptually limited and will not be counted as a concept in this rubric. A fraction <u>can</u> be a part of a whole but there are also many other types of fractions, such as 7/4, which cannot be considered a part of a single whole. A fraction can also express a relation between two quantities where neither is part of the other.

3. Number of Facts and Procedures (Facts)
Record number of facts and/or procedures related to the problem or topic.

Count one point for each piece of information or each procedure that the student demonstrates or explains. "Facts" include defininitions (such as the definition of a fraction as part of a whole) and other statements from memory such as "6 is half of 12." Typical procedures include: adding two fractions, finding equivalent fractions, drawing a circular representation of a fraction.

- Do not give additional points for multiple instances of the same procedure; e. g., if the student computes 20 different fractions equivalent to 1/2, this still counts as knowing one procedure. Or if the student draws 20 different circle and rectangle representations of the same fraction, this would count as knowledge about two procedures: knowing how to draw an area and a circle representation. A number line representation would count as an extra point.
- See attached examples.

4. Misconceptions/Errors (MIS)
Award score points according to this scheme:
1 - one or more serious misconceptions
2 - one or more factual or procedural errors
3 - no errors or misconceptions

A serious misconception is a conceptual error such as believing that there are only a few fractions between 0 and 1.

Factual errors are mistakes in definitions or in descriptions of procedures.

Procedural errors are mistakes in carrying out procedures.

- See attached examples.

5. Integration/Argumentation (INT)
How well does the student integrate facts, procedures, principles, and concepts to develop a coherent problem solving strategy or conceptual argument?
(1-5 point global rating: 1 = no integration, 5 = highest level of integration)
For example, students who successfully integrate graphics and text to make a point should receive at least a "2." Review anchor papers to determine other score points.

General Impression of Quality Content (GICQ)

Anchor Papers

GICQ = Low 2

Explanation Task

Imagine a person from a television station has asked you to give a demonstration on TV. You will be on a show to help other students learn about math. You are asked to explain everything fifth grade students should know about fractions.

Below are some questions you should try to answer. These are questions that students in the TV audience will ask you.

For each question you should draw as many pictures as you can to show what you mean. Then write down what you would say about your pictures on TV. Use as many words and pictures as you need.

What is a fraction?

a fraction is a pis of somthing to sho a difirenl.

Why are there two numbers in a fraction?

Becuse if a pie is cut in to 8 parts then the butter number is 8. If you tack 2 frme the 8 then the fraction will b $\frac{2}{8}$.

3

How many fractions are there between 0 and 1?

1

How many fractions are equal to 1/2?

$\frac{3}{6}, \frac{4}{8}$

What other important ideas should students know about fractions? Show how you would explain these ideas. Use as many pictures and words as you need. If you need more space, continue on to the next page.

4 [redacted]

GICQ = Low 3

Explanation Task

Imagine a person from a television station has asked you to give a demonstration on TV. You will be on a show to help other students learn about math. You are asked to explain everything fifth grade students should know about fractions.

Below are some questions you should try to answer. These are questions that students in the TV audience will ask you.

For each question you should draw as many pictures as you can to show what you mean. Then write down what you would say about your pictures on TV. Use as many words and pictures as you need.

What is a fraction?

a fraction is a way of telling you how many has taken from a number for exp you take - 1 out of 2

Why are there two numbers in a fraction?

because they both mean somthing and you need it to tell what has been taken out the top number means how many you have taken out the bottom number means how many you have.

How many fractions are there between 0 and 1? $\frac{1}{2}$

How many fractions are equal to 1/2? alot because all you have to do is cut Something into even parts + take away half.

$\frac{2}{4}$, $\frac{5}{10}$, and more

What other important ideas should students know about fractions? Show how you would explain these ideas. Use as many pictures and words as you need. If you need more space, continue on to the next page.

GICQ = Low 4

Explanation Task

Imagine a person from a television station has asked you to give a demonstration on TV. You will be on a show to help other students learn about math. You are asked to explain everything fifth grade students should know about fractions.

Below are some questions you should try to answer. These are questions that students in the TV audience will ask you.

For each question you should draw as many pictures as you can to show what you mean. Then write down what you would say about your pictures on TV. Use as many words and pictures as you need.

What is a fraction?

A Fraction is a math problem that has two numbers. A fraction looks like this $\frac{1}{4}$ that means that if you had a candy bar and cut it into 4 pieces and ate 3 of them you would have $\frac{1}{4}$ left It would look like this: you ate the colored part.

Why are there two numbers in a fraction? the bottom number shows how many pieces some thing is cut into. The top number tells how many pieces are taken away.

waJ

How many fractions are there between 0 and 1?

$\frac{1}{4}$ exc $\frac{1}{2}$ $\frac{3}{4}$ $\frac{2}{3}$ $\frac{1}{3}$ $\frac{2}{4}$ exc.

How many fractions are equal to 1/2?

1

$\frac{1}{2}$ $\frac{2}{4}$ $\frac{4}{8}$ $\frac{8}{16}$ $\frac{16}{32}$ $\frac{32}{64}$ $\frac{64}{128}$ $\frac{128}{256}$ $\frac{256}{512}$ exc.

What other important ideas should students know about fractions? Show how you would explain these ideas. Use as many pictures and words as you need. If you need more space, continue on to the next page.

8 [redacted]

GICQ = Low 5

Explanation Task

Imagine a person from a television station has asked you to give a demonstration on TV. You will be on a show to help other students learn about math. You are asked to explain everything fifth grade students should know about fractions.

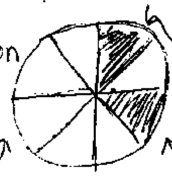
Below are some questions you should try to answer. These are questions that students in the TV audience will ask you.

For each question you should draw as many pictures as you can to show what you mean. Then write down what you would say about your pictures on TV. Use as many words and pictures as you need.

What is a fraction?

It's two numbers thats help you know how many have been eaten or how many things are left.

Example!



I ate this one

this one

this one

$\frac{3}{8ths}$ So I ate 3 out of 8ths

So a fraction tells you how many and things are there and how many have been taken away.

Why are there two numbers in a fraction?

Numerator ← $\frac{1}{3}$

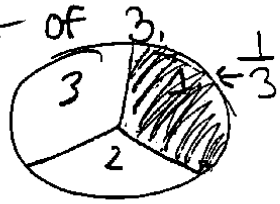
Demnator ←

The top number is ~~the~~ numerator

The bottom number is the Dnominator.

and the slash means out of so it's 1 out of 3.

Denometer shows how many things you're dealing with the numerator shows ~~have~~ how many things you take away.



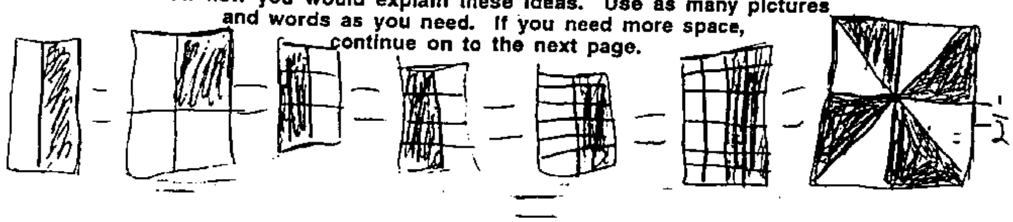
How many fractions are there between 0 and 1?

as many as you want or ~~for~~ ever.

How many fractions are equal to 1/2?

$\frac{2}{4}$ $\frac{4}{8}$ $\frac{6}{12}$ $\frac{8}{16}$ $\frac{9}{18}$ $\frac{10}{20}$ $\frac{20}{40}$ $\frac{30}{60}$ and on

What other important ideas should students know about fractions? Show how you would explain these ideas. Use as many pictures and words as you need. If you need more space, continue on to the next page.



How many fractions are there between 0 and 1?

Infinity, because 1 whole can be broken into as many peices as you like, you can use any number But as you keep deviding it, it will go on forever, there is no other way to say it except infinity. There is no last number.

Which is larger $\frac{2}{5}$ of a pizza or $\frac{2}{4}$ of a pizza?
Draw a picture that shows your answer is correct.
Explain your picture.

$\frac{2}{4}$ $\frac{2}{5}$

$\frac{2}{4}$ is larger. $\frac{2}{4}$ is equal to one half, and one half is large. Fifths are smaler. There is only one thing I do not understand. If the pizza that was cut into fifths, was a large pizza, and the one that was cut into forths was a small, $\frac{2}{5}$ is bigger.

What other important ideas should students know about fractions? Show how you would explain these ideas. Use as many pictures and words as you need. If you need more space, continue on to the next page.

They should know The shortcut way to multiply 2 fractions Like

Ok you take the 5 and the 45 and divide

$\frac{3}{\cancel{5}_1} \times \frac{\cancel{45}^9}{1} = \frac{27}{1} = 27$

Why are there two numbers in a fraction?

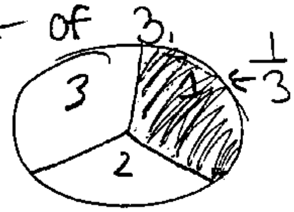
Numerator ← $\frac{1}{3}$

Denominator ←

The top number is the numerator

The bottom number is the Dnominator. and the slash means out of so it's 1 out of 3.

Denometer Shows how many things you're dealing with the numerater Shows ~~have~~ how many things you take away.

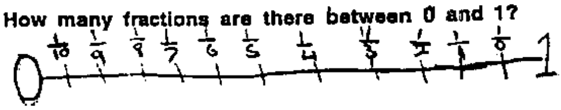


$\frac{1}{2}+\frac{1}{6}=\frac{4}{6}$

Explain how your picture shows the answer.

If you have $\frac{1}{2}$ and you add $\frac{1}{6}$ you will not get $\frac{4}{6}$ if you add $1+1=$ you will get 2 and if you add $2+6=$ you will get 8 so the answer is not $\frac{4}{6}$ it is $\frac{2}{8}$

How many fractions are there between 0 and 1?



There are 10 fractions

Examples of Factual/Procedural Errors

Explain why these pictures show $\frac{3}{6}$.

$\frac{3}{6}$ is equal to $\frac{1}{2}$ and $\frac{6}{12}=\frac{12}{24}=\frac{24}{36}\frac{36}{48}=$ etc.

How many fractions are equal to 1/2?

$\frac{800}{900}$ $\frac{400}{800}$ $\frac{300}{600}$ $\frac{200}{225}$ $\frac{10,000}{20,000}$

$\frac{1000}{8000}$ $\frac{500}{1000}$ $\frac{2000}{4000}$ $\frac{50}{100}$

$\frac{100}{200}$

Integration (INT)/Argumentation (ARG)

Anchor Papers

20611

ARG/INT = Low 2

Explanation Task

Imagine a person from a television station has asked you to give a demonstration on TV. You will be on a show to help other students learn about math. You are asked to explain everything fifth grade students should know about fractions.

Below are some questions you should try to answer. These are questions that students in the TV audience will ask you.

For each question you should draw as many pictures as you can to show what you mean. Then write down what you would say about your pictures on TV. Use as many words and pictures as you need.

What is a fraction? a fracton is somthing that has theBottom number shows how many are in all and the top number show how many have Bean used

Why are there two numbers in a fraction?

How many fractions are there between 0 and 1?

How many fractions are equal to 1/2?

What other important ideas should students know about fractions?
Show how you would explain these ideas. Use as many pictures
and words as you need. If you need more space,
continue on to the next page.

15

ARG/INT = Low 3

Explanation Task

Imagine a person from a television station has asked you to give a demonstration on TV. You will be on a show to help other students learn about math. You are asked to explain everything fifth grade students should know about fractions.

Below are some questions you should try to answer. These are questions that students in the TV audience will ask you.

For each question you should draw as many pictures as you can to show what you mean. Then write down what you would say about your pictures on TV. Use as many words and pictures as you need.

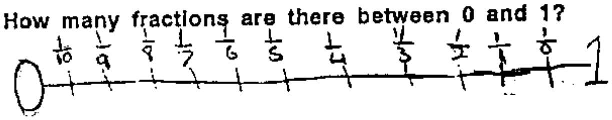
What is a fraction?

a fraction is 2 numbers one one the bottom and the other one the top. If you add a frction to a nothe one Like this $\frac{1}{2} \cdot \frac{1}{2} =$ 1hole our you can say $\frac{1}{2}+\frac{1}{2}=\frac{2}{2}$ then are the same you do not add the bottom ones together

Why are there two numbers in a fraction?

Because if you only have one number then it would be called a number. So you need somthing Like this $\frac{2}{3}$ to mack a Fraction.

How many fractions are there between 0 and 1?



There are 10 fractions

How many fractions are equal to 1/2?

5 fractions are equal to 1/2

$\frac{5}{10}, \frac{4}{8}, \frac{3}{6}, \frac{2}{4}, \frac{1}{2}$

What other important ideas should students know about fractions? Show how you would explain these ideas. Use as many pictures and words as you need. If you need more space, continue on to the next page.

Fracitons are very important.
Youll need them in school at work
and alots of other places.
that is a grate thang to know.

ARG/INT = Low 4

Explanation Task

Imagine a person from a television station has asked you to give a demonstration on TV. You will be on a show to help other students learn about math. You are asked to explain everything fifth grade students should know about fractions.

Below are some questions you should try to answer. These are questions that students in the TV audience will ask you.

For each question you should draw as many pictures as you can to show what you mean. Then write down what you would say about your pictures on TV. Use as many words and pictures as you need.

What is a fraction?

a fraction is a way of telling you hom many has takun from a number for exp you take-$\frac{1}{2}$ out of

Why are there two numbers in a fraction?

because they both mean somthing and you need it to tell what has been taken out the top number means how many you have taken out the bottom number means how many you have.

How many fractions are there between 0 and 1?

as many as you want or for ever.

How many fractions are equal to 1/2?

$\frac{2}{4}$ $\frac{4}{8}$ $\frac{6}{12}$ $\frac{8}{16}$ $\frac{9}{18}$ $\frac{10}{20}$ $\frac{20}{40}$ $\frac{30}{60}$ and on

What other important ideas should students know about fractions? Show how you would explain these ideas. Use as many pictures and words as you need. If you need more space, continue on to the next page.

That between numbers you can go for ever.

ARG/INT = Low 5

Explanation Task

Imagine a person from a television station has asked you to give a demonstration on TV. You will be on a show to help other students learn about math. You are asked to explain everything fifth grade students should know about fractions.

Below are some questions you should try to answer. These are questions that students in the TV audience will ask you.

For each question you should draw as many pictures as you can to show what you mean. Then write down what you would say about your pictures on TV. Use as many words and pictures as you need.

What is a fraction?

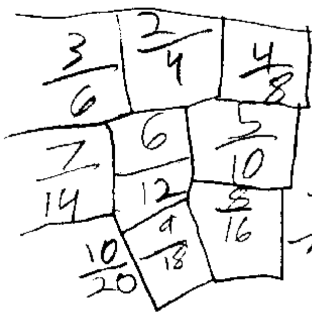
Fractions are measurements that can be added. Fractions are like a puzzle. Fractions aren't hole numbers.

Why are there two numbers in a fraction?

The bottom number is like
how many puzzle peices there
are. The
peices top is how many
are put together.

How many fractions are there between 0 and 1? Lots, any number that is not all the way shaded in.

22/100 1/2 2/3 3/5 300/10000 6/20



How many fractions are equal to 1/2? Nonstop numbers. Any numbers in 2 equal parts. One side is shaded the other side should not.

What other important ideas should students know about fractions? Show how you would explain these ideas. Use as many pictures and words as you need. If you need more space, continue on to the next page.

Fractions are all number on top of each other. They can be added, subtracted, multiplied, and divided. Fractions are fun to work with (Even at school.)

Appendix G

Classroom Observation Data

Tables G1 and G2 show the number of each type of student and teacher action observed, and report each action as a percentage of total observed behaviors. Nine classrooms in each instruction group were observed for one or two class periods.

Table G1
Classroom Observations of Teacher Roles

	Principles		Activity	
	Count	Percent of Total	Count	Percent of Total
Teacher Role	422		448	
Explain	59	13.98	53	11.83
Question	95	22.51	76	16.94
Answer	20	4.74	28	6.25
Goal Relevance	114	27.01	107	23.88
Direct Ongoing Work	16	3.79	18	4.02
Correct/Grade	2	0.47	9	2.01
Test	0	0.00	5	1.12
Reflect Dialogue	5	1.19	3	0.67
Facilitate Discussion	23	5.45	19	4.24
Feedback/Monitor/Help	31	7.35	47	10.49
Confer/Diagnose	2	0.47	7	1.56
Joint Problem-solve	0	0.00	4	0.89
Read to Students	0	0.00	3	0.67
Manage/Oversee	22	5.21	17	3.80
Control/Discipline	20	4.74	10	2.23
Review	12	2.84	39	8.71
Total	422	100.00	422	100.00

Table G2
Classroom Observations of Student Behaviors

	Principles		Activity	
	Count	Percent of Total	Count	Percent of Total
How Working				
Teacher-led	115	63.19	112	59.89
Small Group or Independent	67	36.81	75	40.11
Total	182	100	187	100
Quality of Student Responses				
Level of Processing				
Low/Rote	37	26.06	37	28.24
Medium	88	61.97	82	62.60
High/Elaborate	17	11.97	12	9.16
Total	142	100	131	100
Product Type				
Repeat/Copy	8	11.43	9	11.54
Select	0	0.00	7	8.97
Construct	62	88.57	62	79.49
Total	70	100	78	100
Student Focus				
Very High	38	25.00	53	35.33
High	63	41.45	55	36.67
Some	37	24.34	39	26.00
Low	14	9.21	3	2.00
Total	152		150	
Appropriate Behavior				
Almost All Appropriate	96	63.16	102	67.55
75% Are	21	13.82	36	23.84
50% Are	21	13.82	11	7.29
25% Are	10	6.58	2	1.33
Almost None Appropriate	4	2.63	0	0.00
Total	152	100	151	100
Difficulty with the Task				
75% Have Difficulty	3	1.97	1	0.69
50% Have Difficulty	23	15.13	15	10.27
25% Have Difficulty	42	27.63	46	31.51
Almost No Difficulty	84	55.26	84	57.53
Total	152	100	146	100
Resources in Use				
Textual	141		195	
Teacher-Made	18	12.76	10	5.13
Hands-On	58		35	
Teacher-Made	13	22.41	3	8.57