PARTITIONING NORMAL POPULATIONS
WITH RESPECT TO A CONTROL

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ABSTRACT

Tong (1969) describes a two-stage procedure for partitioning k normal populations with respect to a control, the populations having a common but unknown variance. This problem can be solved for unknown and unequal variances with a modification of a procedure given by Dudewicz and Dalal (1975). This paper derives the appropriate probability equation to be solved so that a correct decision is made with probability at least P^*. Tables of the required h values are given for k=1(1)24, P^*=.75, .90, .95, .99 and n_0=2(1)15(5)30 where n_0 is the sample size in the first stage of the two-stage procedure.

1. INTRODUCTION

Using an indifference zone formulation, Tong (1969) describes ranking and selection procedures for partitioning k normal populations with respect to a control. The situation may be described as follows:
Let $\Pi_0, \Pi_1, \ldots, \Pi_k$ be k+1 normal populations having common variance σ^2 and unknown means $\mu_0, \mu_1, \ldots, \mu_k$, respectively. Let $\delta^* > 0$ be an arbitrary but fixed constant and consider

$$\Omega_B = \{\Pi_i: \mu_i < \mu_0 - \delta^*\}$$

$$\Omega_I = \{\Pi_i: \mu_0 - \delta^* < \mu_i < \mu_0 + \delta^*\}$$

$$\Omega_G = \{\Pi_i: \mu_i \geq \mu_0 + \delta^*\}.$$

After observations are taken the populations $\Pi_1, \ldots, \Pi_k$ are to be assigned to one of two disjoint sets, say S_B and S_G. A correct decision (CD) is made if every population in Ω_B is assigned to S_B and every population in Ω_G is assigned to S_G. No restriction is put on those populations in Ω_I; the populations in Ω_I are classified correctly with probability one. Let P^* be a preassigned constant such that $2^{-k} < P^* < 1$. The goal is to make a correct decision with probability at least P^*; the problem is to determine the minimum number of observations so that the desired probability guarantee is attained.

Given k normal populations having unknown and unequal variances, Dudewicz and Dalal (1975) derived a two-stage solution to the problem of selecting the normal population having the largest mean. Their solution can be modified to yield a solution to the problem considered by Tong for the case of unequal variances. The purpose of this paper is to provide the appropriate probability equation to be solved as well as the tables needed to carry out the solution.

2. SOLUTION FOR UNKNOWN AND UNEQUAL VARIANCES

We now give a solution for the more general case in which the variances are unknown and not necessarily equal. To attain the desired goal we modify slightly the two-stage procedure of Dudewicz and Dalal (1975). Let $h=h(k, P^*) > 0$ be the unique solution of the equation

$$(1) \quad \int_{-\infty}^{\infty} F_\nu^{k-r}(z+h)(1-F_\nu(z-h))^r f_\nu(z)dz=P^*$$

where $r=k/2$ if k is even and $r=(k-1)/2$ or $r=(k+1)/2$ if k is odd and where F_ν and f_ν are the distribution function and density function of a Student's t distribution with $\nu=n_0-1$ degrees of freedom.

For the first stage, let $X_{i1},\ldots,X_{in_0}$ represent a sample of size n_0 from Π_i and define

$$\overline{X}_i(n_0) = \sum_{j=1}^{n_0} X_{ij}/n_0, \; s_i^2 = \sum_{j=1}^{n_0} (X_{ij} - \overline{X}_i)^2/(n_0-1)$$

$$n_i = \max\left\{n_0+1, \left[\left(\frac{s_i h}{\delta^*}\right)^2\right]\right\}$$

where $[y]$ denotes the smallest integer greater than or equal to y. For the second stage take n_i-n_0 additional observations $X_{in_0+1},\ldots,X_{in_i}$ from Π_i and define $\tilde{X}_i = \sum_{j=1}^{n_i} a_{ij}$. The a_{ij}'s are chosen so that $a_{i1}=\ldots=a_{in_i-1}=c_i$, $a_{in_i}=1-(n_i-1)c_i$, where

$$c_i = \frac{n_i-1 \pm\sqrt{(n_i-1)^2 -(n_i-1)n_i[1-(\delta^*/h)^2/s_i^2]}}{(n_i-1)n_i}$$

Form the sets S_B and S_G as follows:

$$S_B = \{\Pi_i: \tilde{X}_i < \tilde{X}_0\}$$

$$S_G = \{\Pi_i: \tilde{X}_i \geq \tilde{X}_0\}$$

Theorem 1. Let $\Pi_{k-g+1},\ldots,\Pi_k$ be the g populations in Ω_G. We now show that for a given g, the infimum of the probability of a correct decision is

$$(2) \quad \int_{-\infty}^{\infty} F_\nu^{k-g}(z+h)(1-F_\nu(z-h))^g \, f_\nu(z)dz.$$

Proof. The proof of theorem 1 follows closely the proof of Dudewicz and Dalal's theorem 4.1. From the definition of a correct decision it follows that to obtain a lower bound on the probability of a correct decision, the set Ω_I must be empty and thus the remaining k-g populations, namely $\Pi_1,\ldots,\Pi_{k-g}$, are in Ω_B. For this special case we have that

$$(3) \quad P(CD) = P\{\tilde{X}_i<\tilde{X}_0,\ \tilde{X}_j\geq\tilde{X}_0 \ (i=1,\ldots,k-g;\ j=k-g+1,\ldots,k)\}$$

$$= P\{\frac{\tilde{X}_i-\mu_i}{\delta^*/h} < \frac{\tilde{X}_0-\mu_0}{\delta^*/h} + \frac{\mu_0-\mu_i}{\delta^*/h},$$

$$\frac{\tilde{X}_j-\mu_j}{\delta^*/h} \geq \frac{\tilde{X}_0-\mu_0}{\delta^*/h} - \frac{\mu_j-\mu_0}{\delta^*/h} \ (i=1,\ldots,k-g;\ j=k-g+1,\ldots,k)\}$$

$$= \int_{-\infty}^{\infty} F_\nu^{k-g}(z+ \frac{\mu_0-\mu_i}{\delta^*/h})(1-f_\nu(z- \frac{\mu_j-\mu_0}{\delta^*/h}))^g \, f_\nu(z)dz.$$

To establish the last equality in (3) we must show that the random variable

$$Y_i = \frac{\tilde{X}_i-\mu_i}{\delta^*/h}$$

has a Student's-t distribution with $\nu=n_0-1$ degrees of freedom.

Let $w_i^2 = \sum_{j=1}^{n_i} a_{ij}^2$. Given s_i^2,

$$U_i = \frac{\tilde{X}_i-\mu_i}{s_i\, w_i}$$

is normally distributed with mean zero and variance σ_i^2/s_i^2. From the choice of c_i it can be seen that

$$s_i^2 \sum_{j=1}^{n_i} a_{ij}^2 = (\delta^*/h)^2$$

As noted by Dudewicz and Dalal (1975) it can now be readily verified using standard techniques that Y_i has a Student's-t distribution with n_0-1 degrees of freedom.

To complete the proof we observe that for populations in Ω_B, $\mu_0-\mu_i \geq \delta^*$, and for populations in Ω_G, $\mu_i-\mu_0 \geq \delta^*$, and so the infimum of (3) occurs at $\mu_0-\mu_i=\delta^*(i=1,\ldots,k-g)$, and $\mu_j-\mu_0=\delta^*(j=k-g+1,\ldots,k)$.

To obtain the infimum over $\underline{\mu} = (\mu_1,\ldots,\mu_k)$ of the probability of a correct decision, it remains to determine the g which minimizes (2). Tong (1969, p. 1322) proves the following lemma. Let s(z) and S(z) be two real-valued functions. Let η be any positive constant and let λ be any real constant and consider

$$\beta(r) = \int_{-\infty}^{\infty} S^r(\eta z+\lambda)\, S^{k-r}(-\eta z+\lambda)\, f(z)dz$$

and its first difference

$$\Delta\beta(r) = \beta(r+1)-\beta(r)$$

If $S(z)$ is monotonically increasing, $S(z) \leq M$ for some $M > 0$, $s(z) \geq 0$, $\int_{-\infty}^{\infty} s(z)dz<\infty$ and if $s(z)=s(-z)$, then

$$\beta(r) = \beta(k-r),\ r=0,1,\ldots,k$$

and

$$\Delta\beta(r) \leq 0,\ r=0,1,\ldots,[(k-2)/2].$$

It is readily verified that the conditions of the lemma are met for f_ν and F_ν. It follows that the infimum of (2) occurs at $r=k/2$ if k is even; $g=(k+1)/2$ or $g=(k-1)/2$ if k is odd. Thus, the infimum of the probability of a correct decision is given by (1) as was to be shown.

3. CALCULATION OF THE TABLES

Calculation of our tables was accomplished in basically the same manner as the one used by Dudewicz and Dalal (1975). There are some differences though and so we describe our procedure in more detail.

We used 32-point Gauss-Legendre numerical quadrature (see, e.g., Krylov, 1962, pp. 107-111) to evaluate (1), the minimum probability of a correct decision. The necessary weights and evaluation points are given by Stroud and Secrest (1966). Since the integration in (1) is performed over an infinite range, we integrated (1) from $-d_1(n_0)$ to $d_2(n_0)$ where $d_1>0$ and $d_2>0$ were set so that the truncation error was at most 10^{-5}. This is possible by choosing d_1 and d_2 at least as large as the values used by Dudewicz and Dalal. (The actual values of d_1 and d_2 used were taken from another paper by Dudewicz, Ramberg and Chen, 1975.)

The tabled values of h are conservative in the sense that the left-hand side of (1) is at least P^*. Simultaneously, the left-hand side of (1) does not exceed P^* by more than 10^{-3}. The values of h were determined using a search routine based on the fact that (2) is a monotonically increasing function in h.

Due to the wide range of $(-d_1, d_2)$, we split the integral over $(-d_1, d_2)$ into several integrals, performed the Gauss-Legendre numerical quadrature on each, and added the results. We used 26 intervals for $n_0=2$, 22 intervals for $n_0=3$, 20 intervals for $n_0=3(1)14$, and 18 intervals for $n_0=20, 25, 30$.

Evaluation of (1) for the special case k=1, h=.1(.1)5.1, $n_0=2(1)15(5)30$ corresponds to the integral evaluated by Dudewicz and Dalal and serves as a check on our calculations. Our results agree with theirs except that on occasion, our value is .0001 larger than the value given by Dudewicz and Dalal. The reason for the discrepancy is that they used 64 points in their numerical quadrature rather than the 32 points used here. However, their results are not necessarily more accurate than ours since we integrated over a larger number of intervals. Moreover, Dudewicz and Dalal report that their results are correct up to $\pm$ .0001 and so this is a negligible discrepancy.

Numerical Illustration

To illustrate the solution, consider four populations Π_0, Π_1, Π_2 and Π_3 where, unknown to the investigator, the corresponding random variables are $X_0 \sim N(0,1)$, $X_1 \sim N(-2, 1/4)$, $X_2 \sim N(2,4)$ and $X_3 \sim N(1.5, 4)$. Let $\delta^*=1, P^*=.9$ and $n_0 = 10$. Thus, a correct decision assigns Π_2 and Π_3 to S_G and Π_1 to S_B. From the table headed $P^* = .9$ we see from the row k=3 and the column $n_0=10$ that the required value of h is 2.88.

Table 1 gives artificially generated data for the first stage of the two-stage solution. Summary statistics are listed at the bottom of the table. From section 2 we see that $n_0=11$, $n_1=11$, $n_2=38$, and $n_3=13$. Thus, for the second stage, we need one additional observation from both Π_0 and Π_1, 28 observations from Π_3 and 3 observations from Π_4. To compute $\tilde{X}_i$, we may use $c_0=.1207$, $c_1=.1519$, $c_2=.0267$, and $c_3=.0827$. Using the artificially generated data in Table 2 for the second stage, we have $\tilde{X}_0=.3263$, $\tilde{X}_1=-1.563$, $\tilde{X}_2=2.606$, and $\tilde{X}_3=2.759$.

TABLE 1

Artificially generated data for the first stage of the two-stage procedure

i	Π_0	Π_1	Π_2	Π_3
1	-1.10	-.11	6.74	1.36
2	.24	-.88	2.68	2.50
3	.59	-.78	7.40	4.46
4	1.22	-1.20	4.92	4.84
5	.53	-.97	1.62	2.54
6	-.11	-1.45	5.60	1.62
7	1.13	-1.54	5.06	2.80
8	-.25	-1.87	2.48	4.42
9	-1.08	-1.16	3.74	2.30
10	.35	-1.44	1.22	2.92
Σx	1.52	-11.4	41.46	29.76
s^2	.6408	.2411	4.544	1.458

TABLE 2

Artificially generated data for the second stage of the two-stage procedure

i	Π_0	Π_1	Π_2	Π_3
11	-.62	-.325	3.84	.96
12			.08	2.10
13			4.52	5.86
14			2.42	
15			4.42	
16			.56	
17			.84	
18			1.10	
19			7.58	
20			2.74	
21			2.98	
22			6.98	
23			2.40	
24			3.36	
25			-0.92	
26			2.06	
27			.74	
28			1.52	

Values of h for $P^* = .90$

$k \backslash n_0$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	20	25	30
1	6.2	3.09	2.51	2.292	2.18	2.11	2.06	2.03	2.00	1.98	1.97	1.95	1.94	1.93	1.897	1.88	1.87
2	9.1	4.6	3.49	3.11	2.91	2.79	2.711	2.66	2.62	2.581	2.56	2.54	2.52	2.502	2.452	2.43	2.41
3	15.9	5.25	3.93	3.46	3.23	3.09	2.993	2.93	2.88	2.841	2.811	2.79	2.77	2.75	2.692	2.66	2.64
4	22.3	5.84	4.28	3.75	3.48	3.32	3.21	3.14	3.081	3.04	3.01	2.98	2.96	2.94	2.872	2.84	2.814
5	25.4	6.32	4.55	3.95	3.66	3.48	3.37	3.29	3.23	3.18	3.14	3.11	3.09	3.07	2.996	2.96	2.933
6	28.5	6.76	4.79	4.13	3.81	3.62	3.496	3.41	3.35	3.295	3.26	3.23	3.197	3.18	3.102	3.061	3.035
7	31.5	7.16	4.99	4.28	3.94	3.74	3.601	3.51	3.441	3.39	3.35	3.313	3.29	3.262	3.19	3.141	3.114
8	34.6	7.53	5.19	4.42	4.05	3.84	3.70	3.60	3.53	3.473	3.43	3.40	3.37	3.34	3.26	3.213	3.19
9	35.0	7.9	5.35	4.54	4.15	3.93	3.78	3.68	3.601	3.55	3.50	3.461	3.43	3.41	3.32	3.273	3.244
10	37.6	8.21	5.51	4.65	4.25	4.01	3.86	3.75	3.67	3.61	3.562	3.523	3.491	3.47	3.38	3.33	3.30
11	41.0	8.52	5.66	4.75	4.33	4.08	3.92	3.81	3.73	3.67	3.62	3.58	3.55	3.52	3.43	3.38	3.35
12	44.0	8.82	5.79	4.85	4.40	4.15	3.99	3.87	3.79	3.72	3.67	3.63	3.60	3.57	3.473	3.421	3.39
13	47.0	9.1	5.92	4.94	4.48	4.21	4.04	3.92	3.84	3.77	3.72	3.673	3.64	3.61	3.513	3.461	3.43
14	50.0	9.4	6.05	5.02	4.54	4.27	4.091	3.97	3.88	3.813	3.76	3.72	3.68	3.65	3.552	3.50	3.464
15	53.0	9.65	6.16	5.10	4.60	4.32	4.14	4.02	3.93	3.86	3.80	3.76	3.72	3.69	3.59	3.532	3.50
16	56.0	9.91	6.27	5.17	4.66	4.37	4.19	4.06	3.97	3.893	3.84	3.791	3.76	3.722	3.62	3.564	3.53
17	59.0	10.2	6.38	5.24	4.72	4.42	4.23	4.10	4.002	3.93	3.871	3.83	3.79	3.76	3.65	3.593	3.56
18	62.0	10.4	6.48	5.31	4.77	4.47	4.27	4.14	4.04	3.97	3.91	3.86	3.82	3.79	3.68	3.621	3.584
19	65.0	10.64	6.58	5.37	4.82	4.51	4.31	4.171	4.071	4.00	3.94	3.89	3.85	3.82	3.71	3.65	3.61
20	68.0	10.9	6.68	5.44	4.87	4.55	4.35	4.21	4.11	4.03	3.97	3.92	3.88	3.842	3.732	3.672	3.633
21	71.0	11.1	6.77	5.49	4.92	4.59	4.38	4.24	4.14	4.06	4.00	3.95	3.902	3.87	3.76	3.70	3.66
22	74.0	11.32	6.86	5.55	4.96	4.63	4.42	4.27	4.17	4.09	4.02	3.97	3.93	3.893	3.78	3.72	3.68
23	77.0	11.53	6.94	5.61	5.00	4.66	4.45	4.30	4.191	4.11	4.05	4.00	3.952	3.92	3.801	3.74	3.70
24	80.1	11.8	7.03	5.66	5.04	4.70	4.48	4.33	4.22	4.14	4.07	4.02	3.98	3.94	3.822	3.76	3.72

Values of h for $P^* = .95$

$k \backslash n_0$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	20	25	30
1	12.7	4.59	3.497	3.11	2.97	2.793	2.72	2.66	2.62	2.59	2.56	2.54	2.52	2.51	2.46	2.43	2.41
2	25.4	6.54	4.59	3.94	3.63	3.45	3.33	3.25	3.19	3.14	3.10	3.07	3.04	3.018	2.95	2.91	2.89
3	32.0	7.5	5.09	4.33	3.96	3.75	3.61	3.52	3.45	3.39	3.35	3.31	3.29	3.26	3.18	3.14	3.11
4	39.0	8.3	5.496	4.63	4.22	3.98	3.83	3.72	3.64	3.58	3.54	3.493	3.47	3.44	3.35	3.298	3.27
5	45.0	8.9	5.82	4.86	4.41	4.15	3.98	3.87	3.78	3.72	3.67	3.63	3.59	3.56	3.47	3.42	3.39
6	52.0	9.51	6.10	5.06	4.57	4.291	4.12	3.99	3.90	3.84	3.78	3.74	3.697	3.67	3.57	3.52	3.48
7	58.0	10.1	6.35	5.22	4.71	4.41	4.23	4.10	3.997	3.93	3.87	3.83	3.79	3.76	3.65	3.59	3.56
8	64.0	10.6	6.57	5.38	4.83	4.52	4.32	4.19	4.09	4.01	3.95	3.90	3.86	3.83	3.72	3.66	3.63
9	70.0	11.1	6.78	5.51	4.94	4.61	4.41	4.26	4.16	4.08	4.02	3.97	3.93	3.90	3.78	3.72	3.68
10	77.0	11.6	6.97	5.64	5.03	4.70	4.48	4.34	4.23	4.15	4.08	4.03	3.99	3.95	3.84	3.78	3.74
11	83.0	12.0	7.14	5.75	5.12	4.77	4.55	4.40	4.29	4.20	4.14	4.09	4.04	4.01	3.89	3.82	3.78
12	89.0	12.4	7.31	5.86	5.21	4.84	4.62	4.46	4.34	4.26	4.19	4.14	4.09	4.05	3.93	3.86	3.82
13	95.0	12.8	7.47	5.96	5.28	4.91	4.67	4.51	4.39	4.31	4.24	4.18	4.13	4.10	3.97	3.90	3.86
14	101.0	13.2	7.62	6.05	5.35	4.97	4.73	4.56	4.44	4.35	4.28	4.22	4.18	4.14	4.01	3.94	3.90
15	108.0	13.6	7.76	6.14	5.42	5.03	4.78	4.61	4.49	4.39	4.32	4.26	4.21	4.17	4.04	3.97	3.93
16	114.0	14.0	7.895	6.22	5.49	5.08	4.83	4.65	4.53	4.43	4.36	4.30	4.25	4.21	4.08	4.00	3.96
17	120.0	14.3	8.03	6.30	5.55	5.13	4.87	4.69	4.56	4.47	4.39	4.33	4.28	4.24	4.11	4.03	3.99
18	126.0	14.7	8.16	6.38	5.60	5.18	4.91	4.73	4.60	4.50	4.43	4.36	4.31	4.27	4.13	4.06	4.01
19	133.0	15.0	8.28	6.45	5.66	5.22	4.95	4.77	4.64	4.53	4.46	4.39	4.34	4.30	4.16	4.08	4.04
20	139.0	15.4	8.39	6.52	5.71	5.27	4.99	4.80	4.67	4.57	4.49	4.42	4.37	4.33	4.19	4.11	4.06
21	145.0	15.7	8.51	6.59	5.76	5.31	5.03	4.84	4.70	4.60	4.51	4.45	4.40	4.35	4.21	4.13	4.08
22	151.0	16.0	8.62	6.66	5.81	5.35	5.06	4.87	4.73	4.62	4.54	4.48	4.42	4.38	4.23	4.15	4.10
23	157.0	16.3	8.73	6.72	5.86	5.39	5.10	4.90	4.76	4.65	4.57	4.50	4.45	4.40	4.25	4.17	4.12
24	164.0	16.6	8.83	6.78	5.90	5.42	5.13	4.93	4.78	4.68	4.59	4.52	4.47	4.42	4.27	4.19	4.14

Values of h for $P^{*}=.99$

k \ n_0	3	4	5	6	7	8	9	10	11	12	13	14	15	20	25	30
1	10.30	6.40	5.20	4.70	4.31	4.12	4.00	3.92	3.82	3.80	3.72	3.70	3.70	3.60	3.50	3.50
2	14.50	8.00	6.20	5.40	5.00	4.72	4.60	4.42	4.32	4.30	4.20	4.13	4.10	4.00	3.90	3.85
3	16.30	8.70	6.70	5.80	5.30	5.01	4.81	4.70	4.60	4.50	4.42	4.40	4.32	4.20	4.10	4.05
4	17.90	9.30	7.10	6.10	5.60	5.30	5.02	4.90	4.80	4.70	4.60	4.53	4.50	4.33	4.24	4.20
5	19.40	9.80	7.40	6.30	5.80	5.40	5.20	5.01	4.90	4.80	4.72	4.70	4.61	4.44	4.40	4.30
6	20.70	10.30	7.60	6.50	5.90	5.60	5.30	5.20	5.00	4.90	4.83	4.80	4.71	4.53	4.44	4.40
7	21.90	10.60	7.80	6.70	6.10	5.70	5.42	5.30	5.10	5.00	4.92	4.90	4.80	4.61	4.51	4.50
8	23.0	11.00	8.10	6.80	6.20	5.80	5.51	5.32	5.20	5.10	5.00	4.92	4.90	4.70	4.60	4.52
9	25.0	11.30	8.20	7.00	6.30	5.90	5.60	5.40	5.30	5.20	5.10	5.00	4.93	4.74	4.63	4.60
10	26.0	11.70	8.40	7.10	6.40	6.00	5.70	5.50	5.32	5.21	5.12	5.10	5.00	4.80	4.70	4.62
11	27.0	11.90	8.60	7.20	6.50	6.10	5.80	5.60	5.40	5.30	5.20	5.10	5.03	4.83	4.73	4.70
12	28.0	12.20	8.70	7.30	6.60	6.10	5.80	5.60	5.43	5.31	5.22	5.20	5.10	4.90	4.80	4.70
13	28.0	12.50	8.90	7.40	6.60	6.20	5.90	5.70	5.50	5.40	5.30	5.20	5.12	4.92	4.80	4.73
14	29.0	12.70	9.00	7.50	6.70	6.21	5.90	5.70	5.53	5.40	5.31	5.23	5.20	5.00	4.84	4.80
15	30.0	13.00	9.10	7.60	6.80	6.30	6.00	5.80	5.60	5.50	5.40	5.30	5.20	5.00	4.90	4.80
16	31.0	13.20	9.20	7.70	6.80	6.40	6.00	5.80	5.61	5.50	5.40	5.30	5.23	5.01	4.90	4.83
17	32.0	13.40	9.30	7.70	6.90	6.40	6.10	5.82	5.70	5.52	5.42	5.33	5.30	5.04	4.93	4.85
18	33.0	13.66	9.40	7.80	7.00	6.50	6.10	5.90	5.70	5.60	5.50	5.40	5.30	5.10	5.00	4.90
19	33.0	13.90	9.60	7.90	7.00	6.50	6.20	5.90	5.72	5.60	5.50	5.40	5.32	5.10	5.00	4.90
20	34.0	14.10	9.70	7.90	7.10	6.50	6.20	5.93	5.80	5.61	5.51	5.42	5.40	5.12	5.00	4.92
21	35.0	14.20	9.80	8.00	7.10	6.60	6.20	6.00	5.80	5.70	5.53	5.50	5.40	5.14	5.02	4.94
22	36.0	14.40	9.80	8.10	7.20	6.60	6.30	6.00	5.81	5.70	5.60	5.50	5.40	5.20	5.04	5.00
23	37.0	14.60	9.90	8.10	7.20	6.70	6.30	6.02	5.90	5.70	5.60	5.50	5.42	5.20	5.10	5.00
24	37.0	14.80	10.00	8.20	7.30	6.70	6.30	6.10	5.90	5.72	5.61	5.52	5.50	5.21	5.10	5.00

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TABLE 5

Approximate Values of h required for $P^*=.75$

$k \backslash n_0$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	20	25	30
1	2.1	1.4	1.3	1.2	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.0	1.0	1.0
2	4.7	2.6	2.2	2.1	2.0	1.9	1.9	1.8	1.8	1.8	1.8	1.8	1.8	1.8	1.7	1.7	1.7
3	***	3.5	2.6	2.4	2.3	2.2	2.2	2.1	2.1	2.1	2.1	2.1	2.0	2.0	2.0	2.0	2.0
4	***	3.5	2.9	2.7	2.5	2.4	2.4	2.3	2.3	2.3	2.3	2.3	2.2	2.2	2.2	2.2	2.2
5	***	3.9	3.1	2.8	2.7	2.6	2.5	2.5	2.5	2.4	2.4	2.4	2.4	2.4	2.3	2.3	2.3
6	***	4.2	3.3	3.0	2.8	2.7	2.7	2.6	2.6	2.6	2.5	2.5	2.5	2.5	2.4	2.4	2.4
7	***	4.4	3.5	3.1	3.0	2.9	2.8	2.8	2.7	2.7	2.6	2.6	2.6	2.6	2.5	2.5	2.5
8	***	4.7	3.6	3.3	3.1	3.0	2.9	2.8	2.8	2.7	2.7	2.7	2.7	2.7	2.6	2.6	2.6
9	***	4.9	3.8	3.4	3.2	3.0	3.0	2.9	2.9	2.8	2.8	2.8	2.7	2.7	2.7	2.6	2.6
10	***	5.1	3.9	3.5	3.2	3.1	3.0	3.0	2.9	2.9	2.9	2.8	2.8	2.8	2.7	2.7	2.7
11	***	***	4.0	3.6	3.3	3.2	3.1	3.0	3.0	2.9	2.9	2.9	2.9	2.8	2.8	2.8	2.7
12	***	***	4.1	3.6	3.4	3.3	3.2	3.1	3.0	3.0	3.0	2.9	2.9	2.9	2.8	2.8	2.8
13	***	***	4.2	3.7	3.5	3.3	3.2	3.1	3.1	3.0	3.0	3.0	3.0	2.9	2.9	2.8	2.8
14	***	***	4.3	3.8	3.5	3.4	3.3	3.2	3.1	3.1	3.1	3.0	3.0	3.0	2.9	2.9	2.9
15	***	***	4.4	3.9	3.6	3.4	3.3	3.2	3.2	3.1	3.1	3.1	3.0	3.0	3.0	2.9	2.9
16	***	***	4.5	3.9	3.6	3.5	3.4	3.3	3.2	3.2	3.1	3.1	3.1	3.1	3.0	3.0	2.9
17	***	***	4.6	4.0	3.7	3.5	3.4	3.3	3.3	3.2	3.2	3.1	3.1	3.1	3.0	3.0	3.0
18	***	***	4.7	4.0	3.7	3.6	3.5	3.4	3.3	3.2	3.2	3.2	3.1	3.1	3.1	3.0	3.0
19	***	***	4.7	4.1	3.8	3.6	3.5	3.4	3.3	3.3	3.2	3.2	3.2	3.2	3.1	3.0	3.0
20	***	***	4.8	4.1	3.8	3.6	3.5	3.4	3.4	3.3	3.3	3.2	3.2	3.2	3.1	3.1	3.0

TABLE 6

Approximate Values of h for $P^*=.90$

$k \backslash n_0$	3	4	5	6	7	8	9	10	11	12	13	14	15	20	25	30
1	3.1	2.6	2.3	2.2	2.2	2.1	2.1	2.0	2.0	2.0	2.0	2.0	2.0	1.9	1.9	1.9
2	4.6	3.5	3.2	3.0	2.8	2.8	2.7	2.7	2.6	2.6	2.6	2.6	2.6	2.5	2.5	2.5
3	***	4.0	3.5	3.3	3.1	3.0	3.0	2.9	2.9	2.9	2.8	2.8	2.8	2.7	2.7	2.7
4	***	4.3	3.8	3.5	3.4	3.3	3.2	3.1	3.1	3.1	3.0	3.0	3.0	2.9	2.9	2.9
5	***	4.6	4.0	3.7	3.5	3.4	3.3	3.3	3.2	3.2	3.2	3.1	3.1	3.0	3.0	3.0
6	***	4.8	4.2	3.9	3.7	3.5	3.5	3.4	3.3	3.3	3.3	3.2	3.2	3.2	3.0	3.0
7	***	5.0	4.3	4.0	3.8	3.7	3.6	3.5	3.4	3.4	3.4	3.3	3.3	3.2	3.2	3.2
8	***	***	4.5	4.1	3.9	3.7	3.6	3.6	3.5	3.5	3.5	3.4	3.4	3.3	3.3	3.2
9	***	***	4.6	4.2	4.0	3.8	3.7	3.7	3.6	3.5	3.5	3.5	3.5	3.4	3.3	3.3
10	***	***	4.7	4.3	3.1	3.9	3.8	3.7	3.7	3.6	3.6	3.5	3.5	3.4	3.4	3.3
11	***	***	4.8	4.4	4.1	4.0	3.9	3.8	3.7	3.7	3.6	3.6	3.6	3.5	3.4	3.4
12	***	***	4.9	4.4	4.2	4.0	3.9	3.8	3.8	3.7	3.7	3.6	3.6	3.5	3.5	3.4
13	***	***	5.0	4.5	4.3	4.1	4.0	3.9	3.8	3.8	3.7	3.7	3.7	3.6	3.5	3.5
14	***	***	5.1	4.1	4.3	4.1	4.0	3.9	3.9	3.8	3.8	3.7	3.7	3.6	3.5	3.5
15	***	***	5.1	4.6	4.4	4.2	4.1	4.0	3.9	3.8	3.8	3.8	3.7	3.6	3.6	3.5
16	***	***	***	4.7	4.4	4.2	4.1	4.0	3.9	3.9	3.8	3.8	3.8	3.7	3.6	3.6
17	***	***	***	4.8	4.5	4.3	4.1	4.1	4.0	3.9	3.9	3.8	3.8	3.7	3.6	3.6
18	***	***	***	4.8	4.5	4.3	4.2	4.1	4.0	4.0	3.9	3.9	3.8	3.7	3.7	3.6
19	***	***	***	4.9	4.6	4.4	4.2	4.1	4.0	4.0	3.9	3.9	3.9	3.8	3.7	3.7
20	***	***	***	4.9	4.6	4.4	4.3	4.2	4.1	4.0	4.0	3.9	3.9	3.8	3.7	3.7

TABLE 7

Approximate Values of h for $P^*=.95$

$k \diagdown n_0$	5	6	7	8	9	10	11	12	13	14	15	20	25	30
1	3.2	3.0	2.8	2.8	2.7	2.7	2.6	2.6	2.6	2.6	2.6	2.5	2.5	2.5
2	4.0	3.7	3.5	3.4	3.3	3.2	3.2	3.1	3.1	3.1	3.1	3.0	3.0	2.9
3	4.4	4.0	3.8	3.7	3.6	3.5	3.4	3.4	3.4	3.3	3.3	3.2	3.2	3.2
4	4.7	4.3	4.0	3.9	3.8	3.7	3.6	3.6	3.5	3.5	3.5	3.4	3.3	3.3
5	4.9	4.5	4.2	4.0	3.9	3.8	3.8	3.7	3.7	3.6	3.6	3.5	3.5	3.4
6	5.1	4.6	4.3	4.2	4.0	3.9	3.9	3.8	3.8	3.7	3.7	3.6	3.6	3.5
7	***	4.8	4.5	4.3	4.1	4.0	4.0	3.9	3.9	3.8	3.8	3.7	3.6	3.6
8	***	4.9	4.6	4.4	4.2	4.1	4.1	4.0	3.9	3.9	3.9	3.8	3.7	3.7
9	***	5.0	4.7	4.5	4.3	4.2	4.1	4.1	4.0	4.0	3.9	3.8	3.8	3.7
10	***	5.1	4.7	4.5	4.4	4.3	4.2	4.1	4.1	4.0	4.0	3.9	3.8	3.8
11	***	***	4.8	4.6	4.4	4.3	4.2	4.2	4.1	4.1	4.0	3.9	3.9	3.8
12	***	***	4.9	4.7	4.5	4.4	4.3	4.2	4.2	4.1	4.1	4.0	3.9	3.9
13	***	***	5.0	4.7	4.6	4.4	4.4	4.3	4.2	4.2	4.1	4.0	3.9	3.9
14	***	***	5.0	4.8	4.6	4.5	4.4	4.3	4.3	4.2	4.2	4.1	4.0	3.9
15	***	***	5.1	4.8	4.7	4.5	4.4	4.4	4.3	4.3	4.2	4.1	4.0	4.0
16	***	***	5.1	4.9	4.7	4.6	4.5	4.4	4.3	4.3	4.3	4.1	4.0	4.0
17	***	***	***	4.9	4.7	4.6	4.5	4.4	4.4	4.3	4.3	4.2	4.1	4.0
18	***	***	***	5.0	4.8	4.6	4.5	4.5	4.4	4.4	4.3	4.2	4.1	4.1
19	***	***	***	5.0	4.8	4.7	4.6	4.5	4.4	4.4	4.3	4.2	4.1	4.1
20	***	***	***	5.0	4.8	4.7	4.6	4.5	4.5	4.4	4.4	4.2	4.2	4.1