TRANSLATION AMONG SYMBOLIC REPRESENTATION IN PROBLEM SOLVING: RESULTS OF PRELIMINARY STUDY

Richard J. Shavelson
Noreen M. Webb
Michal Shemesh
and
Jin W. Yang

CSE Report No. 270

Center for the Study of Evaluation
Graduate School of Education
University of California, Los Angeles

1987

The project presented, or reported herein, was performed pursuant to a Grant from the Office of Educational Research and Improvement/Department of Education (OERI/ED). However, the opinions expressed herein do not necessarily reflect the position or policy of the OERI/ED and no official endorsement by the OERI/ED should be inferred.

Translation among Symbolic Representations in Problem-Solving

Richard J. Shavelson
University of California, Santa Barbara

Noreen M. Webb
University of California, Los Angeles
Center for Research on Evaluation, Standards, and Student Testing

Michal Shemesh
Technion, Haifa

Jin-Wen Yang
University of California, Los Angeles
Center for Research on Evaluation, Standards, and Student Testing

Introduction

Problem solving in many subject-matter domains often requires the problem solver to transform a problem from its original symbolic representation (e.g., words) into an alternative symbolic representation (e.g., iconic, mathematical) in order to arrive at a solution (Clement, Lockhead, & Monk, 1980; Hooper, 1981; Nesher, 1982; Shavelson, 1981; Shavelson & Salomon, 1985). Consider, for example, the following word problem: "Start with one beaker of red solution and one beaker of water. Place a teaspoon of red solution from the first beaker into the second beaker. Then place a teaspoon of liquid from the second beaker into the first beaker. Is the amount of red solution in the first beaker equal to the amount of water in the second beaker?" The verbal presentation usually leads to a logical mistake. Recognizing that the word problem can be transformed into an algebraic representation leads readily to the correct solution.

For a variety of reasons, to be discussed below, students may have difficulty translating from the symbolic representation of the problem to another symbolic representation that leads to a solution. Or, they may be able to translate among certain symbolic representations more easily than among others. For the purpose of assessing subject matter knowledge, tests that fail to take these possibilities into account may produce distorted estimates of students' achievement. Problems on typical achievement tests often require certain kinds of translation (for example, from words to algebra or graphs) and not others. Yet, students may be able to solve the same problems when presented in an alternative symbolic form. If so, typical achievement tests would underestimate these students' subject matter knowledge.

No research to date has systematically investigated the relationships among the symbolic representations of problems given to students to solve, the representations that students use to solve the problem, and the accuracy of their solutions. This study examined some aspects of these relationships. Specifically, it systematically examined students' problem-solving processes and performance as they solve essentially the same problems presented in a variety of symbolic representations. The purposes of the study were (a) to describe the mental representations that students used while solving problems, (b) to determine the kinds of translation (if any) that took place, (c) to determine the accuracy of their mental representations and translation, and (d) to link the accuracy of their representations to problem-solving performance.

Taxonomies of Symbolic Representations

Most of the research on symbolic representations and translation among them has focused on describing categories of representations and on defining translation. Lesh, Post, and Behr (in press), for example, described five contexts for solving mathematical problems: real scripts which organize knowledge around "real world events that serve as models for interpreting and solving other kinds of problem situations" (p. 6), manipulative models (e.g., Cuisinaire rods) which contain elements that correspond to relationships and operations in many everyday situations, spoken languages including special languages such as logic, and written symbols, which in addition to English, may involve specialized sentences such as $x + 3 = 6$.

Hooper (1981) developed a more fine-grained typology for some categories: equations--mathematical formalizations that represent precise relations between things; graphs--simultaneous displays both of the general relationship between variables and the specific relationships between particular values of a set of variables; tables--an intermediate representation between equations and graphs that provides information about the relations among values of a set of variables; pictures--displays of specific instances, making the problem concrete, and diagrams--visual displays of the general characteristics of a problem that demonstrate abstract concepts such as Venn diagrams, pie charts, flow charts. We adapted Hooper's taxonomy to representations of science concepts. We used equations, graphs, tables, and diagrams from Hooper's taxonomy. We also used a fifth representation, numerical exercises.

No one claims that the symbol systems just mentioned are single or pure symbol systems (see Lesh, 1981). Diagrams, for example, may not be single or pure symbol systems. Schematic drawings of electric wiring, pie charts, flow charts, and Venn diagrams can all be considered diagrams but they use somewhat different symbols (circles, squares, numbers, arrows) and even the symbols that they have in common may have different meanings (compare a circle in a flow chart with a circle in a pie chart). At present, because no one knows whether these differences influence students' problem-solving processes, clarifying the type of symbolic representation used in any particular study is very important.

Definition of Translation

One comprehensive definition and description of the translation process between symbol systems comes from Lesh and colleagues. Lesh (in press) defined "translation among representations in mathematics as a problem-solving process of: (1) translating from the 'given situation' to a mathematical model, (2) transforming the model so that desired results are apparent, and (3) translating the model based result back to the original problem situation to see if it is helpful and makes sense" (p. 1). Lesh, Post, and Behr (in press, p. 8) identify five steps in the translation process, corresponding to modeling a mathematical problem: (1) simplifying the problem by ignoring irrelevant information, (2) mapping between the givens and the 'model,' (3) transforming the properties of the model to arrive at a result, (4) translating the result back to the givens, and (5) evaluating the fit of the result to the givens. Lesh (in press, p. 2) provides the following example in which the same problem can be translated from words into algebraic or geometric representations: "A boat, traveling upstream on a river, takes two hours to reach its destination eight miles away. The return trip downstream takes one hour and twenty minutes. What is the speed of the river current?"

Lesh argues that "The ability to do these translations are significant factor influencing both mathematical learning and problem-solving performance" (Lesh et al., in press, p. 7). Indeed, students able to solve mathematical problems do so by representing the problems not in a single symbol system, but in several systems, each corresponding to different parts of a word problem (Lesh, Landau, and Hamilton, 1983).

Most students, however, not only have difficulty understan ding word problems and pencil and paper computations, they lack an understanding about models and languages needed to represent and manipulate ideas in problems (Behr et al., 1985; Post, 1986). To diagnose these difficulties, Lesh et al. (in press, p. 8) recommended presenting an "idea in one representational mode and asking the student to render the same idea in another mode. Then, if the diagnostic questions indicate unusual difficulties with one of the [symbolic representations]... other [representations]... can be used to strengthen or bypass it."

We agree with Lesh that instruction should encourage translation among symbolic forms. Given the fact that much current instruction does not focus on this concern or even deal with a considerable variety of symbolic representations, an important first step is to determine the kinds of translation that students engage in spontaneously and the effects on the accuracy of their solutions to problems. The results have implications not only for instruction but also for achievement testing. Typical tests may not measure the full range of students' problem-solving abilities. Achievement tests present problems in one dominant form: verbal, multiple-choice (often word) problems. Not all students have experience with such representations, especially if other representations have been taught with greater frequency, or if this representation has not been taught specifically. Hence, switching from the usual verbal multiple-choice word problem to other representations might reveal knowledge that otherwise would be judged absent.

<u>Empirical Studies of Translation</u>

Although the notion that students have difficulty translating among symbolic forms is commonplace, few studies have systematically investigated the extent of students' difficulties. Clement, Lockhead, and Monk (1980) showed that even advanced students have difficulty translating a word problem described verbally to an equation. For example, only 39% of freshman engineering majors were able to correctly solve the following problem: "Write an equation using the variables C and S to represent the following statement: At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered strudel. Let C represent the number of cheesecakes ordered and let S represent the number of strudels ordered."

(Clement, Lockhead, & Monk, 1980, p. xxx). For the following problem, "Write an equation for the following statement: There are six times as many students as professors at this university. Use S for the number of students and P for the number of professors.", only 63% of 150 calculus students and 43% of 47 non-sciene majors taking college algebra were able to generate a correct algebraic representation (Clement, Lockhead, & Monk, 1980, p. xxx). The most common error was trying to match the words and algebraic symbols too closely (see also Galvin & Bell, 1977; Nesher, 1979; Paige & Simon, 1966). This matching would, for example, produce the equation 6S = P for the students and professor problem. Clement, Lockhead, and Soloway (1980) suggested that students did not have difficulty understanding the wording of the problem, but instead misinterpreted the syntax of the the equation they generated.

In an attempt to discover whether students had difficulty with translations in general, or only from non-algebraic representations into algebra, Hooper (1981) extended the work of Clement et al. by asking calculus and pre-calculus students to generate an equation, a graph, a table, a picture, and a diagram for the student-professor word problem. Hooper found that the difficulties that some students demonstrated in translating from words to algebra generalized to other representations as well: a subset of students could not represent the concept of a variable in any representational mode. Thus, Hooper (1981, p. 31) concluded that "errors in a number of representational domains suggest that the errors observed in the equations represent a fundamental lack of understanding of the relationships between variables and general unfamiliarity with the conventions of all mathematical representations, not only equations. The errors in equations do not, then, simply reflect difficulties in syntactic translation from verbal descriptions to equations or the lack of procedural specificity in equations."

Clement et al.'s (1980) and Hooper's (1981) studies required students to translate from a single, given symbolic representation to specific other(s). While their studies were very important in showing the difficulties that students have in translating among symbolic representations under those conditions, they did not show the kinds of translation that students engage in spontaneously. The study reported here focused on whether students can correctly interpret and solve the same kind of problem given in multiple symbolic forms. The focus of the present study, then, was on the kinds of translation that students engaged in spontaneously when solving problems presented in a variety of symbolic forms.

Determinants of Translation

Although there is no empirical data on the kinds of translation that students engage in spontaneously when presented with prpoblems in several symbolic forms, it possible to generate a number of hypotheses to describe when translation will or will not take place.

No-translation hypothesis. One hypothesis is that the symbolic form of a problem as given will set boundaries on the mental representations that students will use to solve the problem. For example, problems presented in words may most likely or initially lead to verbal representations, whereas problems presented as diagrams may lead to nonverbal, pictorial representations.

Symbolic encoding specificity. A second hypothesis, one that competes with the first, is that the symbolic representations used in instruction may set boundaries within which students learn and remember concepts. This encoding specificity may place restrictions on students' abilities to translate the problem as given into a symbolic representation that admits to a solution. Encoding specificity may characterize students' knowledge especially when concepts are learned initially. Fuller understanding may come with repeated exposure to the material in different contexts and with repeated application of the concepts to different types of problems. With full understanding of the material, multiple symbolic representations of the same concept can be recognized and used to solve problems.

Due to the limitations in depth of coverage and time allocated to important concepts in much instructional contexts, we suspect that students' acquire only partial understanding from many courses. This partial understanding is probably dominated by the verbal and limited range of other symbolic representations used in classroom instruction.

We cannot find research directly related to symbolic encoding specificity. The closest evidence is indirect and comes from studies finding that people acquire different knowledge from different media. The results of Thorndyke and Hayes-Roth's (1982) map-learning study, for example, supported their hypothesis that when learning from a map, people encode global spatial relations, images that can be scanned and measured like a physical map; whereas when learning through navigation, people acquire knowledge about the routes connecting different locations.

Aptitude or preference. A third hypothesis is that students may have a preference or aptitude for a particular symbolic form; this preference will guide the mental representation(s) that students use to solve problems. Furthermore, this aptitude or preference may vary from student to student or from group to group (e.g., cultural background). In this perspective, students' understanding of a subject will depend greatly on the symbolic representations used in instruction, students' preferred symbolic mode of representing that subject, and the fit between instruction and aptitude.

Cronbach and Snow (1977) reviewed research trying to uncover interactions between aptitudes for learning material in different modes of communication (e.g., verbal, figural) and instructional treatments varying the modes of communication. Not only were the results of the studies mixed, but Cronbach and Snow concluded that the studies often focused on the wrong questions or used treatments or aptitudes that were broad or ambiguous. For example, they point out that the aptitudes and instructional treatments used did not always correspond closely: for example, they did not see how graphs, pictures, and diagrammatic schemes would capitalize on spatial reasoning ability (p. 280). Furthermore, they described how some labeling of treatments may be incomplete: "we come to realize that a task is to be characterized not merely by its stimuli and the required responses, but also by the way a good performer processes the information. In classifying spatial material and generalizing a rule, verbal processes can be as important as spatial ones" (Cronbach & Snow, 1977, p. 281). The present study avoids the latter shortcoming by explicitly investigating students' problem-solving processes for problems presented in different symbolic modes.

Task demands. A final hypothesis is that other features of the problem may guide the kinds of translation that take place. For example, students may translate from the symbolic form of the problem to the symbolic representation of the response required. Problems requiring students to draw a picture may lead to representations concerned with imagery, regardless of the whether the problem was initially presented in words, numbers, tables, or some other symbolic form.

The present study could test some of these hypotheses, but not others. For example, we did not control instructional histories of students, so we could not test the encoding specificity hypothesis. Furthermore, we did not have a measure of symbolic aptitude or preference and so could not examine whether instructional history interacted with individual preference. The present study was, however, designed to investigate the kinds of translation that took place and to relate the resulting translation (or lack of it) to various features of the problems (symbolic form, type of response required).

Methodology

Overview

A test of the ideal gas laws, representing five different symbolic representations, was administered to 20 high school students. Students were asked to think aloud as they solved the problems. Transcripts of their protocols were analyzed to determine the kinds of symbolic representations students used while solving those problems. Students' answer sheets were used to assess the accuracy of their responses.

Sample

Twenty students from two eleven-grade classes in a Los Angeles high school participated in this study. The classes were selected to represent two types of classes to vary ability and coverage of the ideal gas laws. Ten of the students were enrolled in an above-average chemistry class that had spent 15 lessons on the laws, and the other ten came from an average physical science class that had spent 10 lessons on the topic. Most students were white; approximately 30% were of Hispanic backgrounds. All students were fluent in English. All students volunteered to participate in the study and were paid $5 for their time.

Test Construction and Structure

The test was designed to systematically vary three factors: (1) three ideal gas laws, (2) two response forms (quantitative, qualitative), and (3) five symbolic representations of the problem description. In general, one item was generated for each combination of these three factors.

First, the test included problems on the three ideal gas laws: Charles' Law (the relationship between temperature and volume of an ideal gas when the pressure is kept constant), Boyle's Law (the relationship between volume and pressure of an ideal gas when its temperature is kept constant), and Gay-Lussac's Law (relationship between pressure and temperature of an ideal gas when the volume is kept constant). No attempt was made to assess students' understanding of each of the specific concepts of pressure, temperature and volume.

Second, there were two types of problems: quantitative and qualitative. Quantitative items typically contained quantitative information in the stem of the item and required students to give a numerical response. Qualitative items did not contain quantitative information and asked students to give non-numerical responses.

Third, problems for each of the laws and type of problem were represented by five different written symbols: words, diagrams, tables, numbers, and graphs.

One item was written to represent each combination of the three factors described above, except qualitative problems in numeric symbolic form. Each student was given a different random sample of 20 problems to solve (the maximum number that students could respond to in 50 minutes). Figure 1 gives examples of each of the five symbolic forms of quantitative problems. Figure 2 gives examples of qualitative problems in each symbolic form (there were no numerical, qualitative problems).

Insert Figures 1 and 2 about here

Test Administration

The test was administered in a one-to-one interview and lasted for 50 to 60 minutes. Each student was interviewed by one of three interviewers. All interviewers followed the same interview procedure, using a general script. The interview started with a short explanation of the "thinking aloud" technique and then a written example of a hypothetical think-aloud protocol was real aloud to the student (see Shavelson, Webb, & Burstein, 1985, for a methodological critique of the think aloud method; see also Bell & Osborne, 1981; Ericsson & Simon, 1980; Osborne & Gilbert, 1980). After the interviewer felt that the student understood what he or she was supposed to do while taking the test, a brief summary of the ideal gas laws was introduced in a written form and the interview started. During the interview students were asked to think aloud while they solved the problem and to write their answers (as well as any other work) in the test booklet. After students gave their answers to the problems, the interviewer asked further questions to determine whether students used other symbolic forms that they did not verbalize while solving the problem. This probing took place whether the student's answer was correct or incorrect. All interviews were audiotaped and transcribed.

Coding of the Interview and Test Sheets

Students' answer sheets were used to code the accuracy of their responses to the problems. In some cases, when students gave several answers to a problem, or changed their answer, the transcript of the interview was used to determine which answer was the student's final response before any probing by the interviewer.

The transcripts of the interviews were analyzed to determine the symbolic form of the mental representations that students used to solve the problems. The following representations were the ones most commonly used: (1) formula (e.g., $P_1V_1/T_1 = P_2V_2/T_2$; partial formulas and incorrect ones were also coded in this category), (2) numerical (any manipulation of numbers, either in the form of an equation or in arithmetic expressions, for example: 300/200 = 1.5/X), (3) diagram (descriptions or drawings of diagrams), (4) verbal (verbal statement of an ideal gas law, for example: "When the pressure is constant, then the volume will increase as the temperature does because that's one of the laws"), (5) image (for example, describing or drawing a picture of moving molecules: "Well, going back to my teacher's original imagery, he has a large glass full of molecules that he shakes up. And he puts it inot a smaller glass and he shook it the same and we saw the pressure increase dramatically"), and (6) graph (students described or drew a graph of the relationship between variables, or refered to a graph they had seen before, "I remember we saw this. I think it was that one"). A category called "other" was used for unique representations that occurred rarely (for example, one student drew arrows in two items to represent increasing pressure and decreasing volume).

Two coders independently coded the accuracy of response and the kind of representation used for a random sample of five items for ten students. The interrater reliability was .89 for accuracy of students' reponses and was .89 for the representations students used.

Because the data source for the students' responses to the problems (the answers they gave) was predominantly their answer sheets and the source of the representations they used was their think-aloud protocol, we are confident that the data on accuracy of responses is independent of the data on students' representations.

Results and Discussion

The analyses focused on four questions: (1) which features of the problem guided students' mental representations, (2) whether some mental representations led to more accurate responses than others, (3) what is the effects of the nominal symbolic form of the problem on the accuracy of students' responses, and (4) what is the effects of the type of problem (quantitative vs. qualitative) on the accuracy of students' responses?

Before presenting the results, a caveat about our use of the term "mental representation" is in order. Students' verbal descriptions of the representations used during problem solving do not necessarily correspond to the symbolic representations they use. Given that their verbal descriptions are probably only approximations of their actual problem-solving processes, we use the term "mental representation" to denote that approximation and to distinguish students' representations from the symbolic representation of the problem as given.

Task Demands and Mental Representations

Were students' mental representations bound by the symbolic form of the problems or were they guided by task demands? This was a test of the "no-translation" hypothesis that predicted that word problems would give rise to verbal representations, numerical exercises would give rise to numerical representations, and so on.

The frequencies of different mental representations for the five symbolic forms are presented in Table 1. The nominal symbolic form of the problem was not nearly as important as the problems' task demands for students' mental representations. In particular, the form of the <u>response</u> required --quantitative or qualitative--seemed to govern students' mental representations. Most of the quantitative problems required a numerical response, or repuired a numerical calculation. Of the quantitative problems, 74% of the word problems evoked formulas or other numerical mental representations, 77% for diagrams, 78% for tables, and 87% for numerical exercises. The one exception was graphs (59%). Students lacked the skill to work quantitatively with graphs.

Insert Table 1 about here

The tendency for students to use mental representations corresponding to response demands, rather than the problem's symbolic form, was also evident for qualitative problems (those requiring a non-numerical response). The predominant mental representation for qualitative problems was verbal: 50% for word problems, 48% for diagrams, 76% for tables, and 56% for graph problems. We speculate that a verbal expression of the relationship among the variables was an important step in all the qualitative problems. Since students were not required to reach a numerical solution, they did not formulate the relationship using formulas or equations. Even when the required response was not verbal (for example, drawing an arrow), verbal statements of the relationship among the variables were probably a prerequisite for solving these problems.

We conclude that students translated from the symbolic form of the problem to a symbolic form corresponding to the response required. Very few students were bound by the nominal form of the problem.

Accuracy of Solutions for Different Mental Representations

The response demand of the problem guided many students' mental representations. Quantitative responses most frequently corresponded to formulas, or numerical representations; qualitative problems most frequently corresponded to verbal representations. Of concern is whether these representations were more likely than others to lead to the correct response.

Data on the accuracy of responses for different mental representations are presented in Table 2. For qualitative problems, verbal representations were no more likely than other representations to lead to the correct solution. Above-average students performed better than average students regardless of the mental representations they used.

Insert Table 2 about here

For quantitative problems, in contrast, some mental representations were more likely to lead to the correct response, and the mental representations most likely to be successful were different for above-average and average students. For above-average students, numeric representations were most likely to lead to the correct response, whereas verbal representations were least likely to. In fact, above-average students who used verbal representations to solve quantitative problems performed no better than average students who used verbal representations. (Although the frequencies are small, the same result holds for students gave no spontaneous evidence of their mental representations when solving quantitative problems).

Although average students often used numeric representations to solve quantitative problems, these representations were less likely than verbal representations to lead to the correct answer. Average students, who were not familiar with the formulas (their teacher had not taught the formulas in class nor did students practice solving quantitative problems using formulas) were likely creating their own numerical relationships among the numbers given in the problem. The high frequency of unidentifiable numerical expressions among students' work supports this speculation.

Accuracy of Representation vs. Accuracy of Response

The interpretation of the results just described implicitly assumes that students who obtained the correct response used a correct representation (e.g., a correct numerical expression instead of an incorrect one). This assumption was correct. Of the 259 items for which students provided both a representation and a response, the accuracy of the representation matched the accuracy of the response in 257 of them, or 99.2%. Of the two cases that did not match, one student used the correct formula but made an arithmetic error and obtained the wrong result. In the other case, the student selected the correct graph, but described the wrong relationship between variables. From these results, then, we can conclude that students who generate a correct representation of the problem (e.g., the correct formula, the correct description of the relationship between variables) will correctly solve the problem.

<u>Relations among Nominal Symbolic Form, Mental Representation, and Accuracy of Response</u>

The problem's nominal symbolic form did not constrain students' mental representations. Students readily translated from the nominal symbolic form to a representation that corresponded more closely to the response required.

The question remains, is there a relationship between nominal symbolic form, mental representation, and accuracy? For example, was translation from word problems to formulas was more effective than, say, translation from diagrams to formulas? Unfortunately, the sample was too small to systematically examine this question; we present only an analysis over all representations. We found few differences across symbolic forms, except for the lower accuracy rates for graph problems (Table 3). This result may be due to several factors, including students' unfamiliarity with graph problems in general, the ambiguity of the instructions given in the problems used in this study (students had difficulty understanding what the problem required), and the complexity of the response students were required to give, particularly in the quantitative problems (calculating and plotting several points, and drawing the curve through the points).

Insert Table 3 about here

Although the results in Table 3 suggest few differences across symbolic forms (except graphs), the interpretation of the results depends on the relative difficulty levels of the problems for different symbolic forms. Although this study attempted to control many features of the problems to produce problems of comparable difficulty, uncontrolled features became apparent after administration of the test. For example, the numerical values of pressure, volume, and temperature varied across symbolic forms of problems. An increase in pressure from 2 to 3 atmospheres may be a more difficult numerical relationship than an increase from 1 to 2 atmospheres. Moreover, some features of the problems were ambiguous. In diagram items, some students were confused as to whether the increase in pressure pertained to the pressure inside or outside the container. Finally, the nominal symbolic form was sometimes confounded with the kind of response required. Consequently, we cannot be sure that the problems were equivalent in difficulty across symbolic forms.

Type of Problem: Quantitative vs. Qualitative

Although the type of problem--quantitative or qualitative--was not considered a central factor when the test was designed, its impact on students' cognition was considerable. Not only did the problem type influence mental representations, the qualitative problems were easier for students than quantitative problems (cf. Table 3).

The most likely explanation is that all students received instruction in the principles underlying the gas laws but students in the average class received no instruction in solving quantitative problems. Although these results should be interpreted with caution, they suggest that the type of problem had a greater impact on students' problem-solving processes and performance than did the nominal symbolic form of the problem.

Concluding Comments

The results provided support for the task demand hypothesis. Contrary to the no-translation hypothesis, the representations that students used to solve a problem depended more on the response required (quantitative vs. qualitative) than on the problem's nominal symbolic form. Many students readily translated from the nominal symbolic form (e.g., words) to a representation that yielded the solution (e.g., a formula).

However, because our test confounded the type of problem stem (quantitative or qualitative) with the type of response, the exact feature of the problem that guided students' use of particular representations is not clear. The stems of problems that required qualitative responses were also phrased in qualitative (non-numerical) terms. It is possible that the form of the problem stem, rather than the response required, guided students' translation. To answer this question, the test would have to systematically vary both the description of the problem and the kind of response required (e.g., quantitative and quantitative, quantitative and qualitative, qualitative and qualitative).

To clarify the role played by the nature of required response in guiding students' mental representations, a test should include a comprehensive set of symbolic forms of problem descriptions and required responses. For example, the responses might include writing tables, or drawing diagrams, pictures, or graphs, as well as responding with numbers or verbal descriptions (as in, for example, Hooper, 1981). Our test required only a small subset of possible response formats. It is possible that we unwittingly selected the representations that students could generate most easily. Perhaps they would have been less successful at representing their answers in drawings or tables.

If a more general test confirms the hypothesis that the symbolic representation of the response required guides the representations that students use to solve problems, this result would have major implications for instruction and testing. To ensure comprehensive understanding of a concept and comprehensive assessment of students' subject-matter knowledge, instruction and achievement tests should systematically vary the symbolic forms of the required responses to problems. Asking for a quantitative response to a problem would show whether a student can use the appropriate formula and manipulate numbers in the correct way.

Asking for a qualitative, verbal response, on the other hand, may shed more light on whether students understand the relationships among variables. In the present study, some students were able to manipulate formulas more easily than they could explain an ideal gas law. Asking students to draw a diagram of the relationship between variables may show still other aspects of their understanding (or lack of it).

This focus on the kind of response the student is asked to give would be a major shift away from the focus on the representation of the information in the problem. Many curricula, especially in mathematics and science, are based on the notion that presenting information to students in multiple symbol systems (pictures, words, manipulatives) will produce fuller understanding of the subject matter than presenting information in one or few symbol systems. If students are required to give the same kind of response (e.g., numerical) regardless of the given symbolic representation of the problem, the variety of symbolic representations may have only limited benefit. The potential may only be realized when the variety of symbolic representations of the given information is linked to a variety of kinds of representations students must produce as solutions.

Finally, future research should investigate the role student aptitude or preference for symbolic representation plays. Students may have preferences for solving problems using particular representations. These preferences may interact with the representations used in instruction, the task demands of the problem, or both.

Author Notes

We would like to thank Pam Latham and Bill Melega and their students for participating in this study, and Tori Gouveia for helping to prepare this manuscript.

References

Anderson, J.R., (1978). Arguments concerning representations for mental imagery. _Psychological Review, 85_, 249-277.

Anderson, J.R., & Bower, G.H. (1973). _Human associative memory._ Washington, D.C. : Hemisphere Press.

American College Testing Program. (1973). _Assessing students on the way to college: Technical report for the ACT Assessment Program._ Iowa City, IA.

Anastasi, A. (1961). Psychological tests: Uses and abuses. _Teachers College Record, 62._ 389-393.

Anastasi, A. (1982). _Psychological Testing, Fifth Edition._ New York: Macmillan Publishing Co.

Baddely, A.D. (1982). Domains of recollection. _Psychological Review, 89,_ 708-729.

Bell, B., & Osborne, R.J. (1981). Interviewing children - a checklist for the I.A.I.interview. _Learning in science project._ University of Waikato, Australia.

Bower, G.H. (1972). Mental imagery and associative learning. In L. Gregg (Ed.), _Cognition in learning and memory._ New York: Wiley.

Brooks, L.R. (1968). Spatial and verbal components of the act of recall. _Canadian Journal of Psychology, 22,_ 349-368.

Campbell, J.T., Crooks, L.A., Mahoney, M.H., & Rock, D.A. (1973). _An investigation of sources of bias in the prediction of job performance: A six-year study._ Princeton, N.J.: Educational Testing Service.

Chapanis, A. (1970). On image and word. In L. Jacobs (Ed.), _The movies as a medium._ New York: Farrar, Straus, & Giroux.

Chase, W.G., & Clark H.H. (1972). Mental operations in the comparison of sentences and pictures. In L. Gregg (Ed.) _Cognition in learning and memory._ New York: Wiley.

Clement, J., Lockhead, J., & Monk, G. (1980). Translating difficulties in learning mathematics. _The American Mathematics Monthly._

Cole, N.S. (1981). Bias in testing. _American Psychologist, 36,_ 1066-1077.

Cole, N.S., & Nitko, A.J. (1981). Measuring program effects. In R.A. Berk (Ed.) , Educational evaluation methodology: The state of the art. Baltimore, Md.: Johns Hopkins University Press.

DiVesta, F.J., & Peverly, S.T. (1984). The effects of encoding variability, processing activity, and rule-example sequences on the transfer of conceptual rules. Journal of Educational Psychology, 76, 108-119.

Eisner, E.W. (1970). Media, expression, and the arts. In G. Salamon and R.E. Snow (Eds.), Commentaries on Research in Instructional Media. Bloomington: Indiana University.

English & English (1966). Dict. of Psychology.

Fitzgerald, W.M., & Vance, I.E. (1970). Other media and systems. National Council of Teachers of Mathematics Yearbook, 33, 110-133.

Gagne, R.M. (1965). The conditions of learning. New York: Holt, Rinehart, and Winston.

Gardner, H. (1983). Frames of mind: the theory of multiple intelligences. New York: Basic Books.

Gardner, H., Howard, V.A., & Perkins, D. (1974). Symbol systems: A philosophical, psychological, and educational investigation. In D.R. Olson (Ed.), Media and Symbols: The Forms of Expression, Communication, and Education. 73rd Yearbook of the National Society for the study of Education. Chicago: University of Chicago Press.

Goodman, N. (1968). The languages of art. Indianapolis: Hackett.

Gross, L. (1974). Modes of communication and the acquistion of symbolic competence. In D.R. Olson (Ed.), Media and Symbols: The Forms of Expression, Communication, and Education. 73rd Yearbook of the National Society for the study of Education. Chicago: University of Chicago Press.

Hooper, K. (1981). Multiple representations within the mathematical domain. Unpublished paper, University of California, Santa Cruz.

Humphreys, L.G., & Taber, T. (1973). Ability factors as a function of advantaged and disadvantaged groups. Jounal of Educational Measurement, 10, 107-115.

Irvine, S.H. (1969). Factor analysis of African abilities and attainments: Constructs across cultures.

Psychological Bulletin, 71, 20-32.

Jay, T.B. (1983). The cognitive approach to computer curseware design and evaluation. Educational Technology, 22-25.

Jensen, A.R. (1968). Social class and verbal learning. In M. Deutsch, I. Katz, & A.R. Jensen (Eds.), Social class, race, and psychological development. New York: Holt, Rinehart & Winston, Ch. 4.

Jensen, A.R. (1980). Bias in mental testing. New York: Free Press.

Johnson-Laird, P.N. (1983). Mental Models. Cambridge, MA: Harvard University Press.

Kallingal, A. (1971). The prediction of grades for black and white students of Michigan State University. Journal of Educational Measurement, 8, 263-265.

Kerr, N.H., & Winograd, E. (1982). Effects of contextual elaboration on face recognition. Memory and Cognition, 10, 603-609.

Kirkpatrick, J.J., Ewen, R.B., Barrett, R.S., & Katzell, R.A. (1968) Testing and fair employment. New York: New York University Press.

Kosslyn, S.M. (1975). Information representation in visual images. Cognitive Psychology, 7, 341-370.

Kosslyn, S.M., & Pomerantz, J.R. (1977). Imagery, propositions, and the form of internal representations. Cognitive Psychology, 9, 52-76.

Lesh, R., Landau, M., & Hamilton, E. (1983). Conceptual models in applied mathematical problem solving research. In R. Lesh & M. Landau (Eds.), Acquisition of mathematical concepts and processes (pp. 263-343). New York: Academic Press.

Linn, R.L. (1975). Test bias and the prediction of grades in law school. Journal of Legal Education, 27, 293-323.

Linn, R.L., & Werts, E.E. (1971). Considerations for studies of test bias. Journal of Educational Measurement, 8, 1-4.

McNemar, Q. (1975). On so-called test bias. American Psychologist, 30, 848-851.

Miller, R.J. (1973). Cross-cultural research in the perception of pictorial materials. Psychological Bulltin, 80, 135-150.

Nesher, P. (1982). Levels of description in the analysis of addition and subtraction word problems. In T.P. Carpenter, J.M. Moser, and T.A. Romberg (Eds.), Addition and subtraction: a cognitive perspective. Hillsdale, NJ: Erlbaum.

Norman, D.A., & Rumelhart, D.E., & the LNR Research Group, (1975). Explorations in cognition. San Francisco: Freeman.

Olmedo, E.L. (1981). Testing linguistic minorities. American Psychologist, 36 1078-1085.

Olson, D.R., & Bialystock, E. Spatial cognition. Hilldale, NJ: Lawrence Erlbaum Associates.

Olson, D.R., & Bruner, J.S. (1974). Learning through experience and learning through media. In D.R. Olson (Ed.), Media and Symbols: The Forms of Expression, Communication, and Education. 73rd Yearbook of the National Society for the study of Education. Chicago: University of Chicago Press.

Ortar, G. (1963). Is a verbal test cross-cultural? Scripta Hierosolymitana (Hebrew University, Jerusalem), 13, 219-235.

Osborne, R.J., & Gilbert, J.K. (1980). A method for the investigation of concept understanding. European journal of science education, 2, 311-321

Paivio, A. (1971). Imagery and verbal processes. New York: Holt, Rinehart, & Winston.

Paivio, A. (1976). Images, propositions, and knowledge. In J.M. Nicholas (Ed.), Images, perception, and knowledge. (The Western Ontario Series in the Philosophy of Science). Dordrecht, Netherlands: Reidel.

Palmer, S.E. (1975). Visual perception and world knowledge: Notes on a model of sensory-congitive interaction. In D.A. Norman & D.E. Rumelhart (Eds.), Explorations in cognition. San Francisco: Freeman.

Pylyshyn, Z.W. (1973). What the mind's eye tells the mind's brain: A critique of mental imagery. Psychological Bulletin, 80, 1-24.

Pylyshyn, Z.W. (1976). Imagery and artifical intelligence. In W. Savage (Ed.), Minnesota studies in Philosophy of science (Vol. 9). Minneapolis: University of Minnesota

Press.

Reed, S.K. (1974). Structural descriptions and the limitations of visual images. Memory and Cognition, 2, 329-336.

Royer, J.M. (1979). Theories of the transfer of learning. Educational Psychologist, 14, 53-69.

Rudner, L.M., Getson, P.R., & Knight, D.L. (1980). Biased item detection techniques. Journal of Educational Statistics, 5, 213-233.

Salomon, G. (1979). Interaction of media, cognition, and learning. San Francisco: Jossey-Bass.

Scheuneman, J. (1979). A method of assessing bias in test items. Journal of Educational Measurement, 16, 143-152.

Segall, M.H., Campbell, D.T., & Herskovits, M.J. (1966). The influence of culture on visual perception. Indianapolis: Bobbs-Merrill.

Shavelson, R.J. (1981). Teaching mathematics: Contributions of cognitive research. Educational Psychologist, 16, 23-44.

Shavelson, R.J., & Solomon, G. (1985). Information technology: tool and teacher of the mind. Educational Researcher, 14,4.

Shepard, R.N. (1978). The mental image. American Psychologist, 33, 125-137.

Smith, S.M. (1982). Enhancement of recall using multiple environmental contexts during learning. Memory and Cognition, 10, 405-412.

Sowell, E. (1974). Another look at materials in elementary school mathematics. School Science and Mathematics, 74, 207-211.

Stanley, J.C. (1971). Predicting college success of the educationally disadvantaged. Science, 171, 640-647.

Stanley, J.C., & Porter, A.C. (1967). Correlation of Scholastic Aptitude Test scores with college grades for Negroes versus whites. Journal of Educational Measurement, 4, 199-218.

Sternberg, R.J. (1984). Toward a triarchic theory of human intelligence. The Behavioral and Brain Sciences, 7, 269-315.

Temp, G. (1971). Test bias: Validity of the SAT for blacks and whites in thirteen integrated institutions. *Journal of Educational Measurement, 8,* 245-251.

Thorndyke, P., & Hayes-Roth, B. (1982). Differences in spatial knowledge acquired from maps and navigation. *Cognitive Psychology, 14,* 560-589.

Tulving, E., & Thomas, D.M. (1973). Encoding specificity and retrieval processess in episodic memory. *Psychological Bulletin, 80,* 352-370.

Vernon, P.E. (1969). *Intelligence and cultural environment.* London: Methuen.

Wiebe, J.H. (1983). Physical models for symbolic representations in arithmetic. *School Science and Mathematics, 83,* 492-502.

Table 1

Frequencies of Symbolic Representation by Type of Problem

Type of Problem		Symbolic Form of Described Representation						No Spontaneous Description[a]	Total
		Formula	Numerical	Diagram	Verbal	Image	Unclassified		
Word	quant.	13	14	3	6	0	0	3	39
	qual.	1	0	5	16	5	1	5	33
Diagram	quant.	8	16	0	4	2	1	10	41
	qual.	1	0	2	11	6	3	17	40
Table	quant.	14	16	0	8	0	0	1	39
	qual.	2	0	0	19	2	2	14	39
Exercise	quant.	14	6	0	3	0	0	0	10
	qual.	-	-	-	-	-	-	-	--
Graph	quant.	5	14	3	8	2	0	3	35
	qual.	1	0	6	13	2	1	10	33
Total	quant.	54	66	6	29	4	1	16	176
	qual.	5	0	13	59	15	6	46	145

[a] Students answered the problem immediately without giving any evidence of their mental representation.

Table 2

Frequencies and Percentages of Correct Solutions with Different Symbolic Representations

		Symbolic Form of Described Representation							
Type of Problem	Students	Numeric[a] f	Numeric[a] %	Verbal f	Verbal %	Other[b] f	Other[b] %	None[c] f	None[c] %
Quantitative									
	Above-average	60	83	7	58	6	75	3	60
	Average	14	30	10	59	3	100	7	58
Qualitative									
	Above-average	4	80	27	84	22	96	22	85
	Average	0	--	15	56	7	58	8	40

[a]Formula and numerical.

[b]Diagram, image, and unclassified.

[c]Immediate answer without description of representation.

Table 3

Frequencies and Percentages of Correct Solutions Across Symbolic Form of Problems

	Nominal Symbolic Form of Problem[a]											
	Words		Diagram		Table		Numerical		Graph		Total	
	f	%	f	%	f	%	f	%	f	%	f	%
Quantitative												
Above-average	16	76	21	95	17	85	12	86	10	50	76	78
Average	9	53	8	42	8	44	3	33	6	40	34	44
Qualitative												
Above-average	20	95	19	83	22	96	--[b]		14	74	75	87
Average	8	69	9	53	9	56	--[b]		4	29	30	51

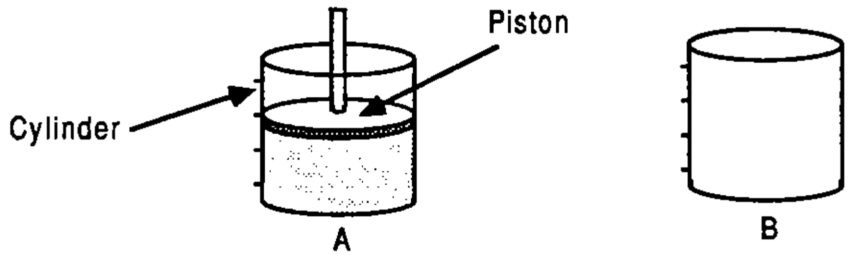
[a]Sample for this analysis includes all students, whether or not they gave spontaneous representations while solving the problem.

[b]Not included on the test.

Word Problem

200 liters of a gas, under pressure of 1 atm, were compressed until the pressure was 4 atm. What will the new volume of the gas be, assuming that there is no change in temperature?

Diagram



V = 4 liters
P = 1 atm. (outside presure)
T = 25°c

v =
P = 2 atm
T = 25°c

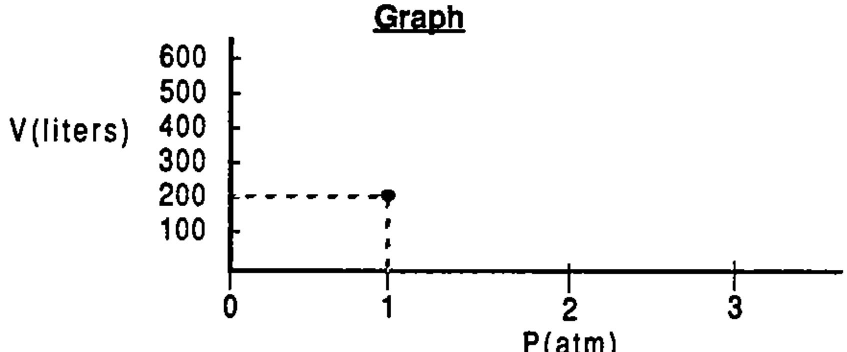
Please draw the exact position of the piston in cylinder B.

Table

T(°c)	P(atm)	V(liters)
25	1	80
25	4	__

Please complete the table by inserting the correct value of V.

Graph



Please complete the graph, assuming a constant temperature.

Numbers

Initial State	Final State
T_1 = 25°c	T_2 = 25°c
V_1 = 120 liters	V_2 = ?
P_1 = 1 atm	P_2 = 4 atm

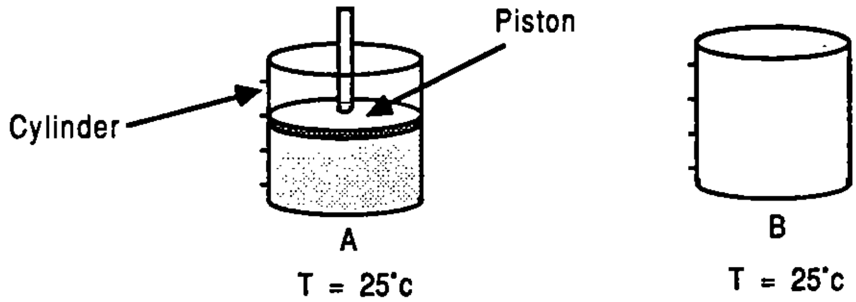
Calculate V_2 in the final state.

Figure 1. Example of quantitative problems for each symbolic form (Boyle's Law).

Word Problem

Assuming gas temperature is kept constant, if the pressure is increased, what happens to the volume of the gas?

Diagram



Please draw the piston in cylinder B, after the outside pressure has been increased.

Table

T(°c)	P(atm)	V(liters)
25	1	200
25	↑	---

Please complete the table by inserting an arrow:

↑ = increase

↓ = decrease

= = the same

Graph

Draw a line that shows the relationship between the volume (V) and the pressure (P) of ideal gas, at a constant temperature:

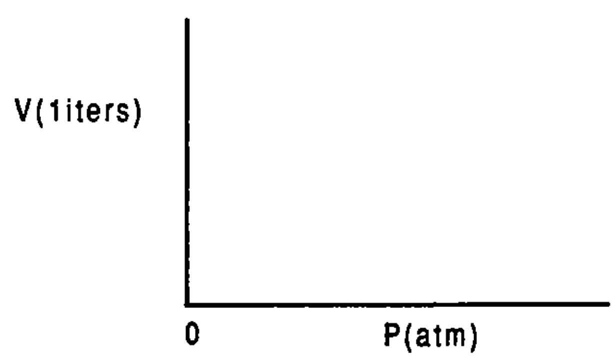


Figure2. Example qualitative problems for each symbolic form (Boyle's Law).