Quantifying the Characteristics of Knowledge Structure Representations: A Lattice-Theoretic Framework

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Abstract

This work shows how lattice theory can be used to develop quantitative measures of selected characteristics of knowledge structure representations, and how these measures can be used to assess individual persons' knowledge structure representations in a classroom setting. For a given set of concepts, a knowledge structure can be described by the present or absent connections in the set of all possible pairwise connections between concepts. Under this description, the set of all possible knowledge structure representations for a given set of concepts are the elements of a complemented, distributive lattice ordered by set inclusion. Measures are developed to assess the dissimilarity between two knowledge structure representations, the local complexity of a concept in a knowledge structure, and the global complexity of a knowledge structure. The effectiveness of these measures in assessing the changes in students' knowledge structure representations in an introductory statistics course is examined using data from Ju (1989).

Introduction

The ability to quantitatively assess individuals' knowledge has important implications for educational measurement, diagnostic assessment, and test development. Snow and Lohman (1989) stress the importance of knowledge structure assessment to the educational process:

> Students build up vast structures of particularized knowledge, both declarative and procedural, over their educational years... Such knowledge is often partial, incomplete, or incorrect in idiosyncratic ways. It is also often tied to particular situations. And it can be brought into new learning in ways that distort the new learning, as well as in ways that allow new learning to complete or correct or the supplant the old. The improvement in knowledge assessment would seem to depend generally on diagnosis, both of prior knowledge and of knowledge in the process of being acquired. Such assessment should include direct attempts to assess how concepts are organized... (p. 304)

Knowledge structure assessment for diagnostic purposes is seen by some as a corrective to the shortcomings of traditional testing practices. Surber (1984) has suggested that traditional testing for diagnosis of misunderstanding suffers from the drawbacks of (a) insensitivity to structure, (b) intrusions and distortions, and (c) errors of omission. Traditional tests are often insensitive to the *structure* of a subject being tested, because any given subject "is not merely a collection of lists of concepts. A discipline is recognized as such because of the interrelationships of its concepts" (Surber, 1984, p. 215). Items on a multiple choice test are often inadequate to test a subject's structural knowledge of a given domain. The alternative of short answer or essay questions may also have problems. In some cases, information that was not a part of the instruction may act as an intrusion in the response given, while in others, information may be distorted in the process of giving an answer. In either case, these test formats provide:

> ... no general method of dealing with declarative knowledge errors in terms of scoring or for diagnostic purposes... [T]hese error responses do not lead to a better understanding of what the learner knows. In short, an essay test does not permit the systematic diagnosis of a learner's misunderstanding. (Surber, 1984, p. 215)

Finally, traditional tests often fail to distinguish between a lack of knowledge and erroneous knowledge; in the case of the multiple choice tests, this may be further confounded by the results of guessing. Additionally, since multiple choice items are often weighted equally, there is no way to assess systematically the importance of the missing knowledge. The need to diagnose for errors of omission is essential if this is to be remedied.

The importance of knowledge structure assessment to the instructional design of diagnostic tests has also been addressed. Nitko (1989) argues that eliciting representations of learners' knowledge structures and comparing them to the "canonical" knowledge structures of experts should be an integral part of diagnostic assessment and instructional design:

> A test designer's understanding of the meaning and structure of the knowledge a student brings to the instructional system is important for building diagnostic tests. Tests of prerequisites should focus on these aspects of the preinstructed learner. Frequently, students' everyday understandings of terms and phenomena are at odds with the experts' canonical understandings. These conflicts can interfere with instruction directed toward acquisition of canonical knowledge, unless students' knowledge schema are explicitly addressed in the course of teaching. (p. 461)

While many writers have recognized the importance of knowledge structure assessment (e.g., Glaser, Lesgold, & Lajoie, 1987; Lane, 1991), it is still the case that "cognitive theories about knowledge structures have progressed far ahead of research on methods for their assessment that would be useful in education" (Snow & Lohman, 1989, p. 304).

In recent years a number of methods have been developed to elicit a representation of a subject's structural understanding for a set of concepts. These techniques have used a simplified model to represent structural knowledge as a network of interconnected nodes. In the model's basic form, each node has represented a concept, the meaning of which is determined by its connections to other concepts (nodes) in the network. Using this model, these techniques elicit knowledge structure representations[1] for a set of concepts by requiring a subject to directly manipulate the nodes representing the concepts, and the connections representing the relationships among them.

While these methods have been used successfully to elicit knowledge structure representations in such knowledge domains as statistics (Rogan, 1988; Ju, 1989), the biological sciences (Fisher, Faletti, Thornton, Patterson, Lipson, & Spring, 1988), physics (Jonassen, 1987; Hegarty-Hazel & Prosser, 1991), chemistry (Ruiz-Primo, Schultz, & Shavelson, 1997; Ruiz-Primo, Shavelson, & Schultz, 1997), and geology (Champagne & Klopfer, 1980; Ballstaedt & Mandl, 1985), the analyses of these representations have often been problematic. Most studies have transformed the data into proximity matrices in order to use a scaling method such as hierarchical cluster analysis or multidimensional scaling. When these matrices have been derived by averaging the data across the subjects, group information has been obtained at the expense of losing the characteristics of the individual knowledge structures. Those studies that have not used proximity matrices (e.g., Champagne & Klopfer, 1980) have relied on analytic techniques that are qualitative rather than quantitative.

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[1] What will be referred to as "structural knowledge" in this paper has several different definitions in the literature. Some writers have associated structural knowledge exclusively with declarative knowledge (e.g., Snow & Lohman, 1989, p. 298), while others have proposed structural knowledge as an intermediate type of knowledge that mediates the translation of declarative into procedural knowledge (Jonassen, Beissner, & Yacci, 1993, p. 4). Structural knowledge has also be referred to as cognitive structure, conceptual knowledge, and semantic networks (Jonassen et al., 1993, p. 5). In any event, the *representation* of such structural knowledge is distinct from the actual underlying cognitive model.

This work shows how *lattice theory*[2] can be used to develop quantitative measures of the selected characteristics of knowledge structures, and how these measures can be used to assess individual persons' knowledge structures in a classroom setting.

Preliminaries: The Lattice of Knowledge Structure Representations

Let C be a set of k distinct *concepts*, where k is finite, and let K be the set of all *connections* between concepts from C. Since this set represents the combinations of the k concepts taken two at a time, the number of connections n in the set K is given by $n = \mathrm{C}_2^k = k!/(k-2)!2! = k(k-1)/2$. The set of connections will be denoted by $K = \{c_1, c_2, \ldots, c_n\}$, and the *powerset* of K (i.e., the set of all subsets of K) by $\wp(K)$.

A *knowledge structure representation (KSR) on C* is a pair (C, X), where C is the set of concepts, and $X \in \wp(K)$ is a set of connections that join them. The *set of all possible knowledge structure representations (on C)* is denoted by $(C, \wp(X))$, where C and $\wp(K)$ and are defined as above. Since there are n connections in K, and each connection is either present or absent in any knowledge structure $X \in \wp(K)$, then the total number of KSRs in $(C, \wp(X))$ is equal to 2^n.

It is natural to consider the set of all KSRs as being ordered: That is, one KSR is less than or equal to another KSR ($(C, X) \leq (C, Y)$), when the set of connections for the first KSR is contained in the set of connections for the second KSR ($X \subseteq Y$ for $X, Y \in \wp(K)$). Under this ordering, this set forms a lattice.[3] This characterization of the set of the all possible KSRs as the *knowledge structure representation lattice (KSR lattice)* allows certain properties to be derived.

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[2] Recent use of lattice theory in psychology, measurement, and testing can be seen in Hirtle (1982, 1987), Haertel and Wiley (1993), and Tatsuoka (1990, 1991). General information regarding the theory of partially ordered sets and lattices can be found in Birkhoff (1967), Donnellan (1968), Salii (1988) and Davey and Priestley (1990). Szasz (1962) also presents a proof showing that the ordering of a finite set can be illustrated with a diagram. The use of lattices in computer science and combinatorics is examined in Stanton and White (1986), as well as in Davey and Priestley (1990). Other approaches to some of this material is provided by Bollobás (1986), who focuses his study on the subsets of a finite set; Harary (1972), Harary and Palmer (1973), Palmer (1985), Buckley and Harary (1989) who explore various aspects of graph theory, including random graphs, distance in graphs, and graphical enumeration.

[3] For any two knowledge structures $(C, X), (C, Y) \in (C, \wp(K))$, there a smallest knowledge structure which is at least as large as either, namely $(C, X \cup Y)$, and a largest knowledge structure which is at least as small as either, namely $(C, X \cap Y)$. The knowledge structures $(C, X \cup Y)$ and $(C, X \cap Y)$ are referred to as the *join* and *meet* of the knowledge structures (C, X) and (C, Y).

First, the KSR lattice has both a *bottom* element (denoted $(C,\varnothing)$) and a *top* element (denoted (C, K)). These elements represent the KSR with no connections at all, and the KSR with all possible connections respectively. Furthermore, since the KSR lattice is distributive (see Young, 1993, p. 226-231), each KSR (C, X) has as its complement (C,X'), the unique element such that $(C,X)\cap(C,X')=(C,\varnothing)$ and $(C,X)\cup(C,X')=(C,K)$. This element can be formed by taking $X'=K\setminus X$, the relative complement of the set of connections X, with respect to set of *all* connections K.

As an example, Figure 1 shows the knowledge structure representation lattice generated from a set of three concepts. The number of pairwise connections of these concepts is $n=3!/(3-2)!2! \ =3(3-1)/2=3$, and is denoted by $K=\{c_1,c_2,c_3\}$. Each KSR is represented by a binary vector, with a one used to indicate the presence and a zero, the absence of a connection. Thus, [0 1 1] is the vector representing the absence of the first connection (i.e., c_1), but the presence of the second and third connections (c_2 and c_3).

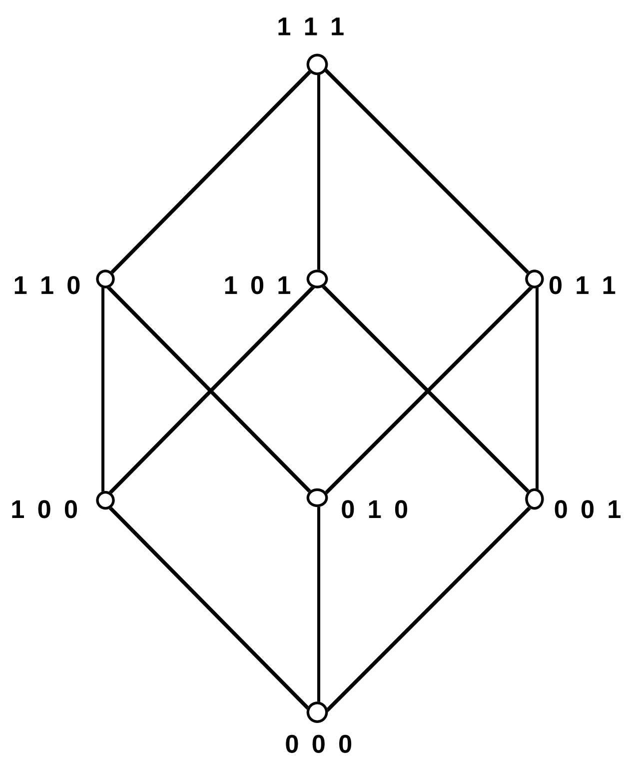


Figure 1. The knowledge structure representation lattice of three connections.

More formally, if $K = \{c_1, c_2, \ldots, c_n\}$ is the set of connections for a KSR lattice, and $Z^n = \bigtimes_{i=1}^{n} \{0,1\}$ denotes the n-fold Cartesian product of the set {0,1}, then we can always define a function $\phi: \wp(K) \rightarrow Z^n$ by $\phi(X) = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ where $\varepsilon_i = 1$ if $c_i \in X$, and $\varepsilon_i = 0$ if $c_i \notin X$ (Davey & Priestly, 1990, p. 20).

The KSR lattice $(C, \wp(X))$ consists of $2^3 = 8$ elements, each representing a different knowledge structure generated from the set of connections. The small circles in this diagram represent the individual KSRs, and the relationships among the KSRs in the lattice are indicated by lines. For a given KSR, lines going up indicate the KSRs that contain it, while lines going down indicate the KSRs that are subsets of it. Note that the bottom of the lattice is the KSR in which none of the concepts are connected (denoted [0 0 0]), while the top of the lattice is the KSR in which all of the concepts are connected (denoted [1 1 1]). By tracing upwards and downwards in the lattice, the intersection and union for any pair of knowledge structures representations can be found. For example, choosing the KSRs denoted by [1 0 1] and [0 1 1], then their intersection will be the element [0 0 1] and their union [1 1 1]. If we consider the diagram of the KSR lattice as a three-dimensional object, then the complement of each element (i.e., knowledge structure) can be thought of as the element in the diagonally opposite corner of the cube.

Measuring the Dissimilarity Between Knowledge Structure Representations

Consider two knowledge structure representations that have been represented as the n-element, binary vectors $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{Y} = (y_1, y_2, \ldots, y_n)$. We can compare these vectors, tabulate the numbers of matched and mismatched elements, and arrange these tabulations in a 2 x 2 frequency table as shown in Figure 2. In this table, a represents the frequency of 1-1 matches, b is the frequency of 1-0 matches, and so forth. Given such an arrangement of the frequencies, a variety of similarity and dissimilarity measures for binary data may be defined (Romesburg, 1984).

Three often used measures are the *simple matching coefficient* (*SM*), the *binary squared Euclidean dissimilarity* (*BSED*) and the *mean character difference* (*MCD*). The simple matching coefficient is defined as $SM = (a + b)/n$, and represents the percent of perfect agreement to be found between the two vectors. The *MCD* and *BSED* are both dissimilarities or distances, and are defined as

		Connections in KSR Y		
		Present 1	Absent 0	
Connections in KSR X	Present 1	1-1 a Co-occurrences	1-0 b Mismatch	a + b
	Absent 0	0-1 c Mismatch	0-0 d Conjoint Absences	c + d
		a + c	b + d	n = a + b + c + d

Figure 2. 2 x 2 table for comparing knowledge structure representation vectors.

$MCD = (b + c)/n$ and $BSED = b + c$. Each of these coefficients are related to the simple matching coefficient by $MCD = 1 - SM$ and $BSED = n - n(SM) = N(1 - SM)$. The *MCD* is thus a rescaling of the *BSED* to a dissimilarity from 0 to 1, and of the SM from a similarity to a dissimilarity measure.

Of these different measures, the binary squared Euclidean dissimilarity provides the most natural interpretation of distance in KSR lattice. Since the *BSED* is just the binary version of the squared Euclidean distance, it is a metric (note that Euclidean distance between **X** and **Y**, $d(\mathbf{X},\mathbf{Y}) = \sum_{i=1}^{n} (x_i - y_i)^2 = b + d$ since only the mismatches are counted). The *BSED* can be directly interpreted as the number of "moves" or "steps" that must be made in the knowledge structure representation lattice, in order to go from one knowledge structure representation to another.

For example, in Figure 1, the furthest distance that can be traveled in the lattice is *BSED* = 3. In general, if the lattice has n connections, then the maximum possible distance between two persons' KSRs is *BSED* $= n$, the distance from a KSR (C, X) to its complement (C, X').

Measuring the Complexity of Knowledge Structure Representations

Two complementary approaches will be used to measure the complexity of knowledge structure representations. The first, a local measure of complexity, is the *degree* of a concept: For a given concept in a knowledge structure, this is simply the number of connections coming from it. For each of the k concepts in a knowledge structure representation, this will be a number from 0 to $k - 1$.

The *rank* (Stanton & White, 1986, p. 31) of a KSR uses the position within the KSR lattice as a global measure of complexity: Each KSR in the lattice is ranked by the number of connections contained within it. In Figure 1, for example, the ranks of the knowledge structures in the KSR lattice range from the minimum number of connections (zero) possible for a KSR to the maximum number (three). For a general knowledge structure representation $(C, \wp(X))$, the rank ranges from 0 (i.e., the KSR with no connections) to n (i.e., the completely connected KSR).

Two further points should be noted. First, that the number of KSRs of rank r generated from a set of n connections is equal to the binomial coefficient $C_r^n = n!/r!(n-r)!$. For example, in Figure 1, there are three KSRs with rank $r = 2$, since $C_2^3 = 3$. Second, it should be noted that rank of a KSR is equal to the one-half the sum of the degrees of the concepts in the KSR.

Using Lattice-Theoretic Measures

The classification and comparison of knowledge structure representations. The use of the *BSED* and rank allows for several comparisons to be made. The amount of change that a subject's structural knowledge undergoes throughout a course due to instruction can be easily examined. For example, consider a subject with knowledge structure vector [1 0 1] at the beginning of a course, with a KSR vector [1 1 0] at the end of the course. The dissimilarity between these two KSRs is a $BSED = 2$, but their ranks are both equal to 2. The overall complexity of the KSRs has not changed from the beginning to the end of the course, but two of the connections between pairs of concepts have changed.

Classifications for a group of knowledge structures can be obtained by clustering KSRs using as characteristics the presence or absence of connections. One procedure might be to include one or more *canonical representations* along with those of the subjects. This would allow subjects' KSRs to be compared to several idealized or expert representations for the same set of concepts.

Composite knowledge structure representations. The classification of interest here can be obtained by using cluster analysis to group connections from the knowledge structure representations using as characteristics the presence or absence of a subject having that connection. A dendrogram from this kind of analysis will group together those pairs of connected concepts shared by the greatest number of subjects. By examining the clusters of connected concept pairs, several KSRs may be analyzed.

First, it might be possible to infer from these clusters of connected concept pairs, the central concepts or ideas held by the group. Second, it may be possible to obtain a composite knowledge structure representation for the entire group, by working backwards from the dendrogram based on these central concepts. Finally, if we use groups of subjects who differ on a variable such as "final course grade," we may be able to obtain classifications of their inferred central concepts based on that variable.

Comparisons to a canonical knowledge structure representation. Definitions of the "correctness" of a knowledge structure representation have usually involved qualitative comparisons of a subject's KSR to that of a canonical representation that resulted from the analysis of a set concepts for a domain. When quantitative measures had been used to assess the correctness, they have usually involved a scoring system (Surber, 1984; Rogan, 1988).

Using the lattice-theoretic measures that have been developed, we can now assess the correctness of a subject's knowledge structure representation in terms of its similarity to a canonical representation. Using the binary squared Euclidean dissimilarity, if a subject's KSR being examined is zero "steps" away from the canonical representation, then it will be completely correct with respect to the pattern of concepts that the subject connected. Information as to the correctness of the relations that the subject may have specified is possible only by studying the language the subject used in the subject's knowledge structure representation.

Another possible analysis is to identify subsets of connections within the KSRs, such that each subset represents a key or central concept. The sub-structures generated by these subsets of connections can then be used to assess the "correctness" of a subject's understanding for those central concepts.

An Example of Using Lattice-Theoretic Measures

Method. The effectiveness of these measures in assessing the changes in students' knowledge structures was examined using data taken from Ju (1989). She used her software program MicroCAM to obtain the knowledge structure representations from six students for fourteen statistics concepts (see Table 1) taken from the topics of central tendency and variability. This was done twice:

Table 1

Statistics Concepts Used in MicroCAM Pretest/Posttest Experiment by Ju (1989)

Statistics concept
1 Central tendency
2 Mean
3 Median
4 Mode
5 Parameter
6 Population standard deviation
7 Population variance
8 Range
9 Sample standard deviation
10 Sample variance
11 Semi-interquartile range
12 Statistic
13 Unbiased estimator
14 Variability

before the students had been formally introduced to the concepts, and after the material had been covered in their statistics course. In addition to these representations, the instructor's knowledge structure representation for the concepts was also obtained for use as a canonical representation. Each of the KSRs was coded as a binary vector, and the matrix of the binary squared Euclidean dissimilarities was calculated for all pair-wise comparisons of the KSRs. The ranks of the KSRs, and the degrees of the concepts in the KSRs were also calculated. Finally, the thirteen subjects' knowledge structures were analyzed using this dissimilarity matrix, and the average link clustering algorithm of the SPSS-X™ procedure CLUSTER (SPSS-X™, Release 3.0, 1988).

The classification and comparison of individuals' knowledge structure representations. The binary squared Euclidean dissimilarity matrix presented in Table 2 allows a number of comparisons of knowledge structure representations to be made. We can see that students H and J had the most similar KSRs before instruction ($BSED = 8$), while after instruction, students I and K were most similar ($BSED = 6$). The last row of Table 2 shows how similar each student's KSR was when compared to that of the teacher's, before and after instruction. Finally, the bold-faced diagonal in the lower left part of the table measures the amount of

Table 2

Binary Squared Euclidean Dissimilarity Coefficient Matrix for Pretest/Posttest Data From Ju (1989)

		Pretest						Posttest					
Time	Subject	F	G	H	I	J	K	F	G	H	I	J	K
Pretest	**G**	17											
	H	18	13										
	I	13	12	11									
	J	14	13	8	11								
	K	16	11	14	11	14							
Posttest	**F**	**19**	18	19	18	15	19						
	G	23	**16**	17	22	13	19	8					
	H	25	14	**17**	20	17	15	6	10				
	I	21	16	15	**16**	11	17	12	8	14			
	J	24	17	18	21	**10**	16	9	7	11	11		
	K	21	20	17	20	13	**13**	10	10	12	6	11	
	T	24	25	22	21	18	20	9	13	13	11	12	7

Note. Student KSRs are F through K, while the teacher's canonical KSR is denoted by T; boldface diagonal shows dissimilarity from student's pretest to posttest KSR.

change between each student's pretest and posttest KSRs. The student's knowledge structure representations changed a great deal from pretest to posttest (*Mean* = 15.17, *SD* = 3.19); this change was found to be significant ($t = 11.95$, $df = 5$, $p < .005$).

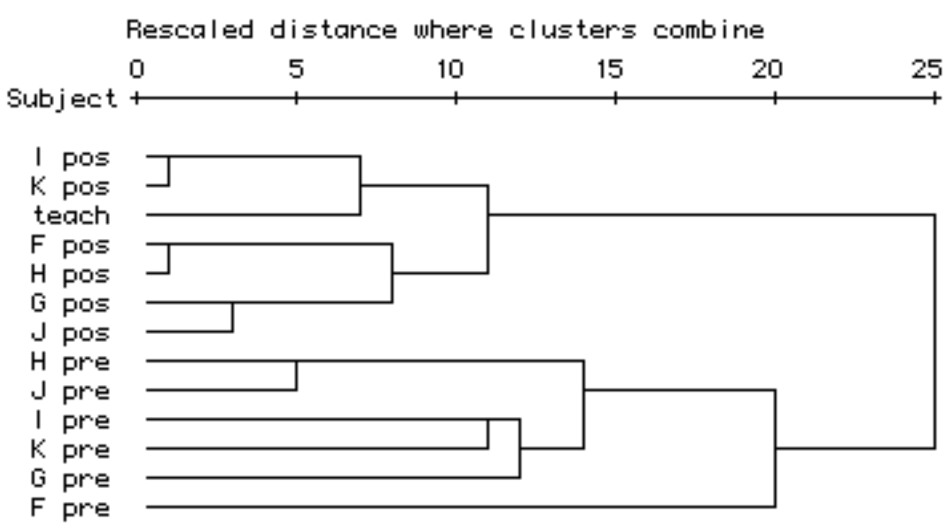
The means and standard deviations for the pre- and posttest dissimilarities between the students' and teacher's KSRs are presented in Table 3. The change in the dissimilarities from pretest to posttest (*Mean* = 10.83, *SD* = 3.19) was found to be significant ($t = 8.32$, $df = 5$, $p < .005$), showing that the students' KSRs were becoming more similar to that of the teacher's.

Table 3

Means and Standard Deviations for Student-Teacher Dissimilarities (Binary Squared Euclidean Dissimilarity)

Time	*N*	Mean BSED	Standard deviation
Pretest	6	21.67	2.58
Posttest	6	10.83	2.40

Figure 3 is the dendrogram for the clustering solution, and shows two main clusters. The first contains the teacher's knowledge structure representation together with the students' posttest structures, while the second cluster contains all of the pretest knowledge structures. The dendrogram gives visual evidence that the similarities were greater within the pretest and posttest groups than across the groups. (If a three cluster solution were preferred, student F's pretest KSR, which was most dissimilar to the other students' knowledge structure representations, would stand alone.)



Teacher's KSR: teach;
Subjects' pretest KSRs: F pre to K pre;
Subjects' posttest KSRs: F post to K post.
Cluster analysis using Average Linkage (Between Groups)

Figure 3. Dendrogram of pretest and posttest knowledge structure representations.

The first method of assessing the complexity of the students' KSRs made use of the lattice-theoretic measure of rank. The ranks of each KSR were found, and the means and standard deviations for the pre- and posttest ranks of the students' and teacher's KSRs were calculated (see Table 4). The change in ranksfrom pretest to posttest (*Mean* = 4.17, *SD* = 1.17) was used as a measure of the change in complexity of the students' knowledge structure representations. Though the increase in complexity as mean change in rank was found to be significant ($t = 7.97$, $df = 5$, $p < .005$), indicating that the students' representations were more complex after instruction, none of the students' KSRs became as complex as that of the teacher (rank = 20). The mean rank of students' knowledge structure representations after instruction was found to be significantly less than the rank of the teacher's knowledge structure representation ($t = -4.60$, $df = 5$, $p < .005$).

A second method for assessing the complexity of students' KSRs examined the mean degree of concept for each of the fourteen concepts presented to the students. The means and standard deviations for the pre- and posttest degrees of concepts, as well as the change in degree for each concept are given in Table 5. Before instruction, the concepts of mean (2), sample standard deviation (9), sample variance (10), unbiased estimator (13), and variability (14) were found to have mean degrees that were significantly lower than the degrees that the teacher had for those concepts (t = -3.16, -17.00, -17.00, -3.37, and -12.85 respectively; df = 5 for all tests; p-values in Table 5). After instruction, the

Table 4

Means and Standard Deviation for Pretest and Posttest Ranks of Student Knowledge Structure Representations

	Rank		
Subject	Pretest	Posttest	Change
F	10	15	5
G	15	19	4
H	14	17	3
I	13	17	4
J	10	16	6
K	10	13	3
Mean	12.00	16.17	4.17
SD	2.28	2.04	1.17

Note. The teacher's knowledge structure representation had rank = 20.

Table 5

Mean and Standard Deviation for the Degree of the Concepts in Subjects' Knowledge Structure Representations

	Teacher	Pretest		Posttest		Change	
Concept	Number	Mean	*SD*	Mean	*SD*	Mean	*SD*
1	4	3.50	.84	3.33*	.52	-.17	.75
2	3	1.67	1.03*	1.33**	.82	-.33	.82
3	1	1.00	.00	1.00	.00	.00	.00
4	1	1.17	.41	1.00	.00	-.17	.41
5	3	1.67	1.51	2.50	.55	.83	1.60
6	2	2.17	1.17	2.50	1.05	.33	.82
7	2	2.17	1.17	2.67	1.21	.50	.84
8	1	1.33	.82	1.00	.00	-.33	.82
9	4	1.17**	.41	2.67**	.82	1.50†	.84
10	4	1.17**	.41	2.67**	.82	1.50†	.84
11	1	.83	.41	1.00	.00	.17	.41
12	4	3.33	1.51	2.33	1.75	-1.00	2.53
13	3	1.33*	1.21	2.17	1.33	.83	1.33
14	7	.50**	1.05	6.17	1.17	4.67†	1.63

* $p < .05$, one-tailed, significantly less than teacher's number of connections.

** $p < .01$, one-tailed, significantly less than teacher's number of connections.

† $p < .01$, one-tailed, significantly greater than zero.

concepts of central tendency (1), mean (2), sample standard deviation (9), and sample variance (10) were the only concepts to have mean degrees (t = -3.14, -5.00, -4.00, and -4.00 respectively; df = 5 for all tests; p-values in Table 5) significantly different from those of the teacher. These results can be interpreted in terms of how complex those concepts were when compared to those of the teacher.

As a measure of which concepts showed the most increase in complexity between pretest and posttest, the mean change in degree for each of the fourteen concepts was calculated. Only the concepts of sample standard deviation (9), and sample variance (10), and variability (14) showed a significant increase in "connectedness" (t = 4.39, 4.39, 7.00; df = 5 for all tests; p-values in Table 5).

Finally, a plot of the ranks of the students' KSRs versus their dissimilarities to the teacher's knowledge structure representation is shown in Figure 4. This plot shows how both of these measures have changed from pretest to posttest, and how the students' knowledge structure representations have become more similar to that of the instructor (indicated by the cross in the circle on the vertical axis).

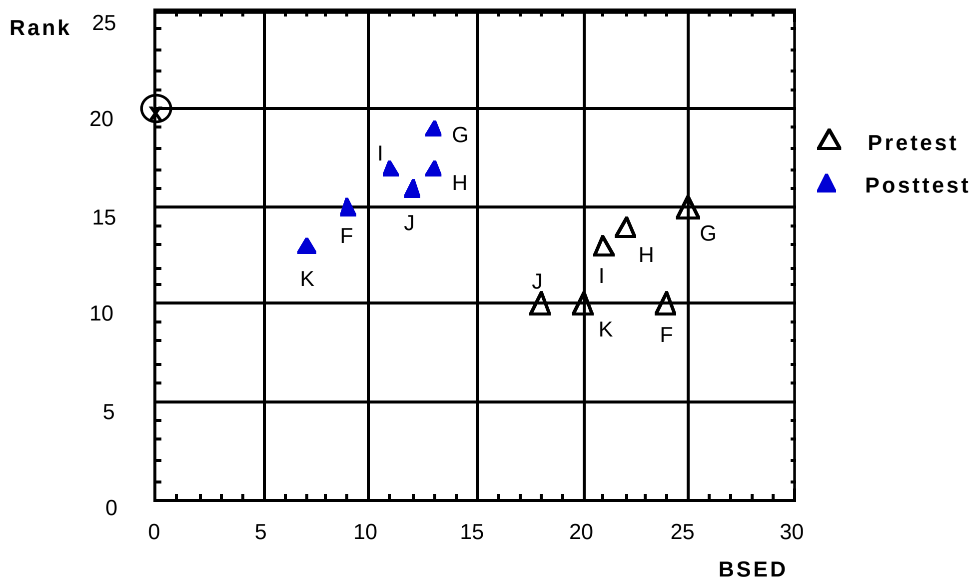
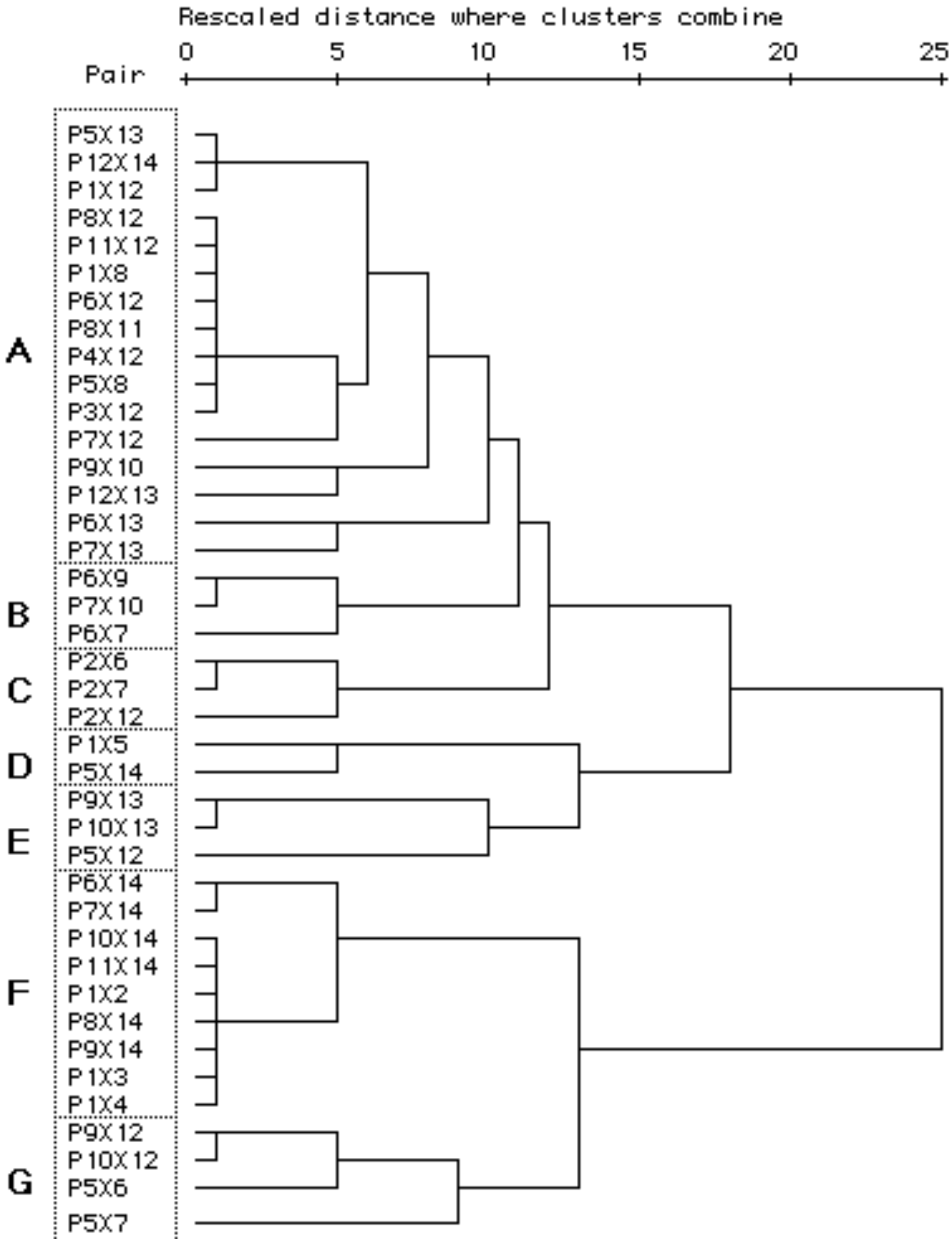


Figure 4. Students' KSR ranks vs. dissimilarities to teacher's KSR.

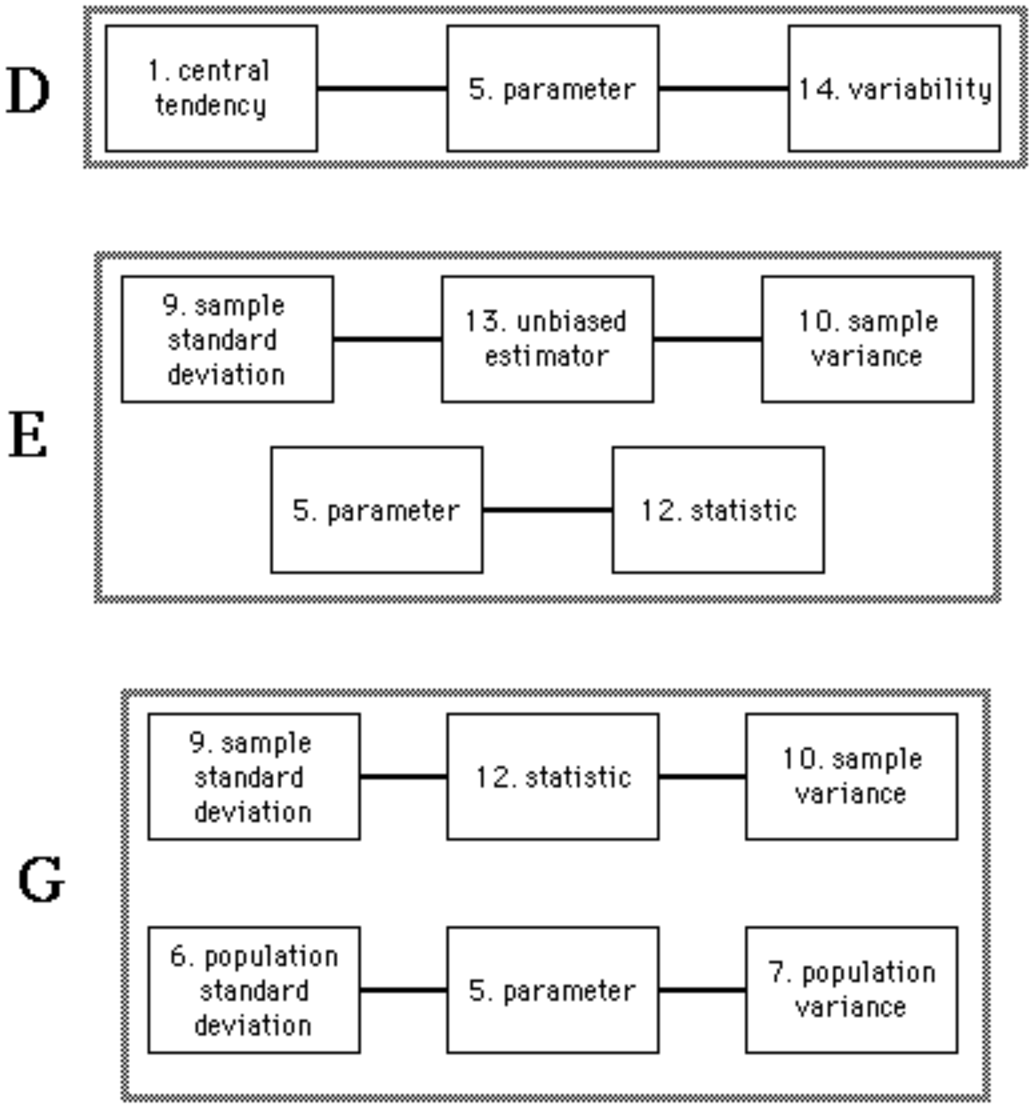
Composite knowledge structure representation. A second cluster analysis was performed on the subjects' posttest knowledge structures representations, to see which connections the subjects tended to group together into larger structures. Rather than clustering the subjects on the basis of the connections they held in common, the connections in the KSRs were grouped on the basis of the proportion of subjects who shared them. The dendrogram for this analysis is shown in Figure 5. Since this was an exploratory analysis, the dendrogram was arbitrarily cut at a seven cluster solution.

This clustering solution was examined as a composite knowledge structure representation, consisting of those clusters of connections forming the central ideas labeled A through G. Given the dendrogram, it was possible to produce a network representation of each of the central concepts in Figure 5. Figure 6 shows the representations of three of those central concepts. Central concept D included the 1-5 and 5-14 connections, and seemed to represent the students' understanding that central tendency (1) and variability (14) are both examples of parameters (5).



Note. “Pair” denotes a connected pair of concepts (e.g., P5X13 refers to the connected pair of concepts 5 (parameter) and 13 (unbiased estimator)). Letters **A** through **G** denote groups of connected pairs of concepts in seven-cluster solution using Average Linkage (Between Groups).

Figure 5. Dendrogram for cluster analysis of students’ posttest connections.



Note. Lines indicate connections for concepts.

Figure 6. Inferred central concepts from cluster analysis of students' posttest connections.

Central concept E was made up of three connections, with the 9-13 and 10-13 connections being grouped separately from the 5-12 connection. Here the central idea seemed to focus on unbiased estimation (13): the sample standard deviation (9) and the sample variance (10) are both examples of unbiased estimators (13); they are statistics (12) that can be used to estimate parameters (5).

Finally, central concept G consisted of the pairs 5-6, 5-7, 9-12, and 10-12. This central concept differentiated between population standard deviation (6) and sample variance (7) as parameters (5), and the sample standard deviation (9) and sample variance (10) as statistics (12).

Central concepts B and F though not pictured, were also readily interpretable. Central concept B was a linking together of the sample and population standard deviations (9, 6) to the sample and population variances (7, 10). Central concept F consisted of two hub-and-spoke groupings, with the mean (2), median (3), and mode (4) being centered on central tendency (1), and the standard deviations (6,9), variances (7,10), range (8), and semi-interquartile range (11) on variability (14).

Central concept C was made up of three connections, with mean (2) forming the hub, and population standard deviation (6), population variance (7), and statistic (12) connecting to it as spokes. This cluster may have to do with how the standard deviation and variance can be defined in terms of deviations from the mean.

Interpretation was more problematic for central concepts A and C. Central concept A was the cluster in the analysis having the greatest number of connections (see Figure 5). Though statistic (12) was the most highly connected concept in this cluster (with seven connections), it was not clear what aspect of the students' knowledge was being captured here.

Comparisons to a canonical knowledge structure representation. A third analysis was performed on a subset of the connections, in order to assess the "correctness" of students' KSR for specific central concepts. The teacher's KSR was used as a canonical knowledge structure representation, and four central concepts were identified by grouping together the connections that made them up. Figure 7 shows the teacher's KSR; the four central concepts of "central tendency," "parameter," "variability," and "estimation" that were derived from this structure are shown in shown in Figure 8.

It should be noted that range (8) was included with the "central tendency" group; this was done to see how the subsequent analyses would handle misconceptions or erroneously added concepts. The dissimilarities for the subjects were once again calculated using only the 19 connections making up the four central concepts, and a cluster analysis of these reduced KSRs performed. The pre- and posttest dissimilarities between the students' and the teacher's KSRs for the four central concepts is presented in Table 6.

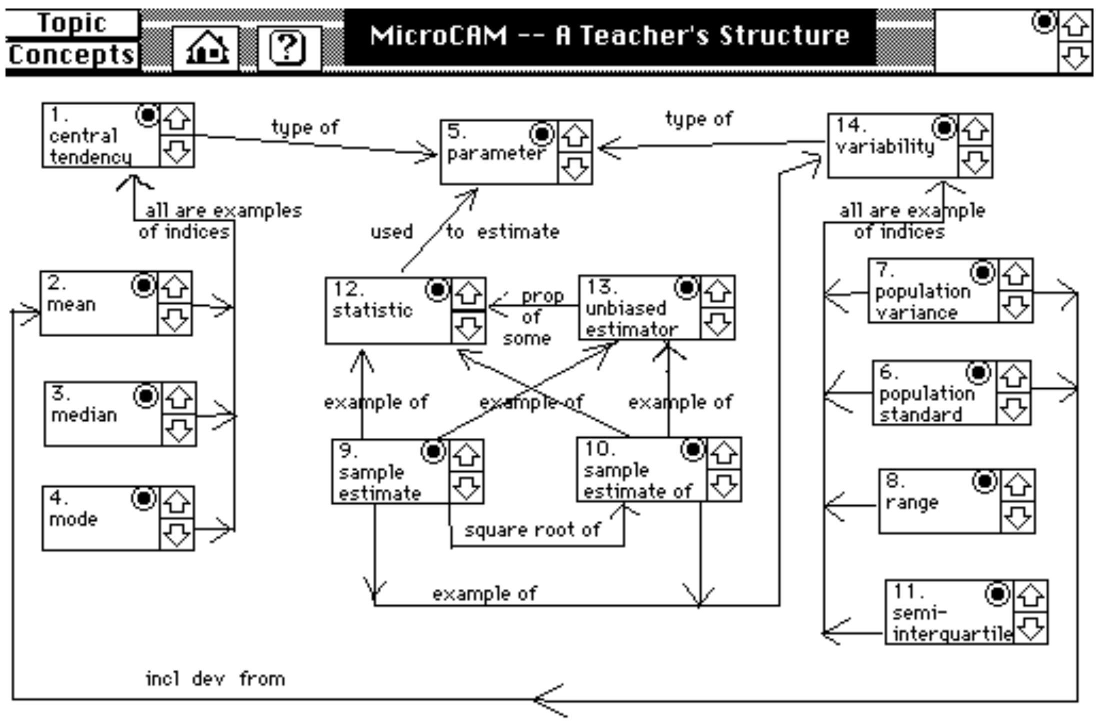
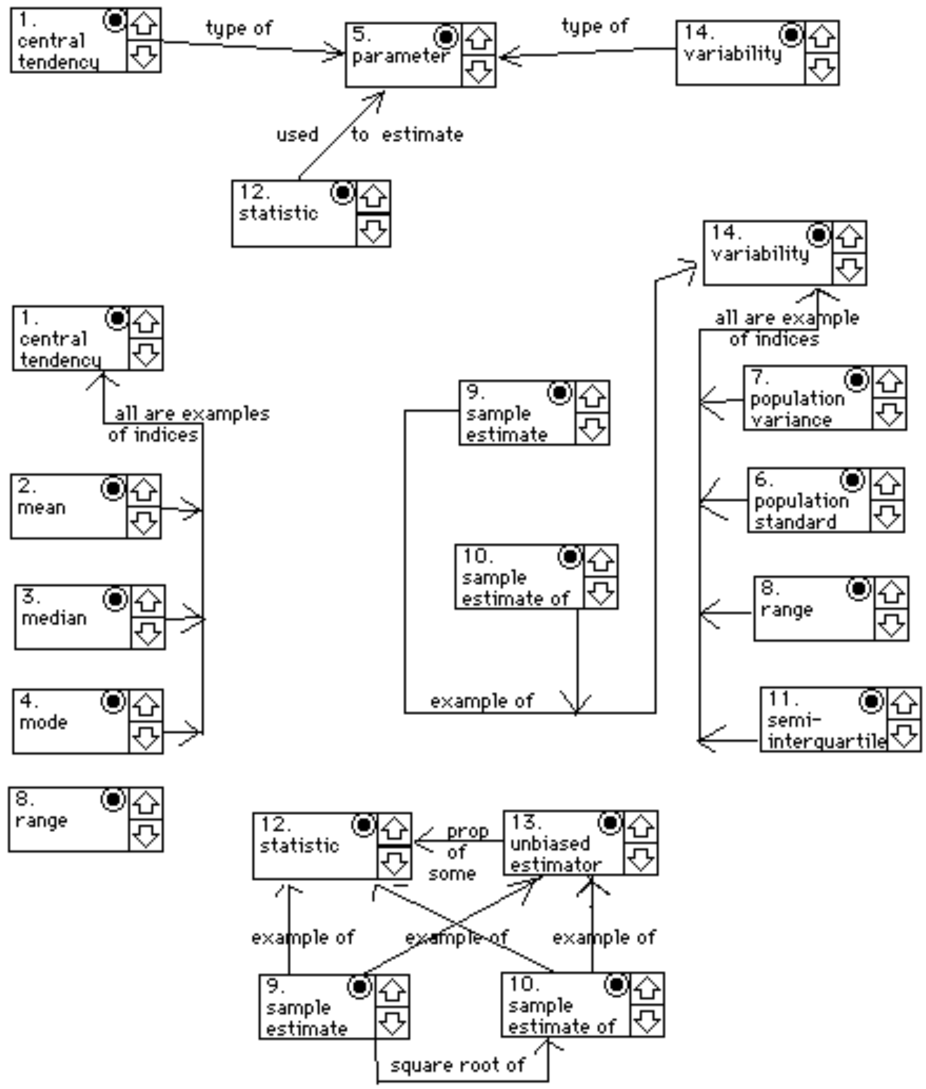


Figure 7. MicroCAM knowledge structure of teacher (Source: Ju, 1989).



Note. Central ideas are clockwise from left, central tendency, parameter, variability, and estimation.

Figure 8. Four central ideas taken from teacher's knowledge structure (After Ju, 1989).

Table 6

Pre- and Post-Instructional Dissimilarity Measures Between Teacher and Subjects for Four Central Concepts

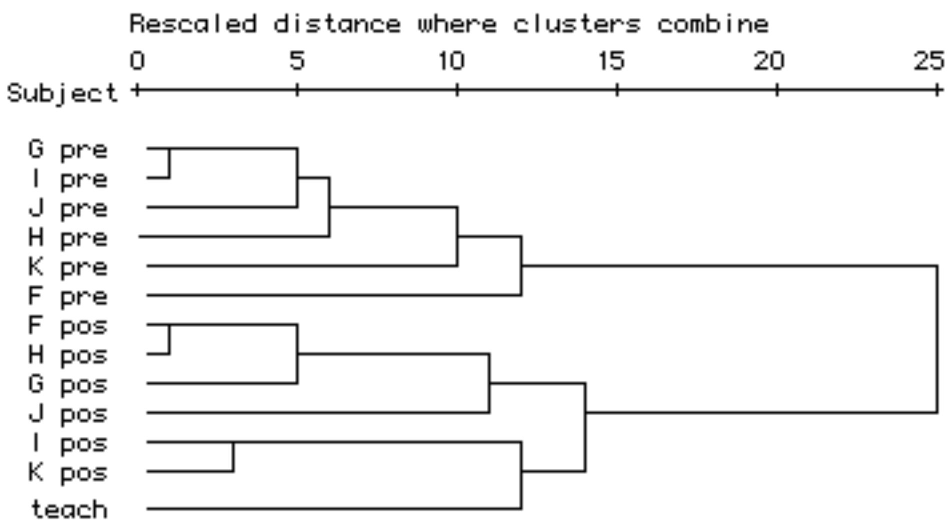
Subject	Central concepts				
	Parameter	Variability	Central tendency	Estimation	BSED
Pretest					
F	3	6	3	6	18
G	2	5	0	6	13
H	2	4	0	6	12
I	2	6	0	6	14
J	2	4	0	6	12
K	3	6	0	4	13
Mean	2.33	5.17	.50	5.67	13.67
SD	.52	.98	1.22	.82	2.25
Posttest					
F	3	0	0	4	7
G	3	0	0	2	5
H	3	0	0	3	6
I	1	0	0	4	5
J	2	2	0	2	6
K	1	0	0	4	5
Mean	2.17	.33	0	3.17	5.67
SD	.98	.82	0	.98	.82
Max	3	6	4	6	19

Of the four central concepts analyzed for the students, only the ideas of "variability" and "estimation" showed a significant increase in similarity to those of the teacher ($t = 6.75$, $df = 5$, $p < .001$ for "variability"; $t = 3.68$, $df = 5$, $p < .01$ for "estimation"): the "correctness" for those concepts could be said to have increased. Overall, the students understanding of "central tendency" (with the exception of Student F) did not improve very much: They essentially had the same idea of central tendency as the teacher did before instruction. The students' gain as the result of instruction was mostly due to an increased understanding of the idea of "variability," where they went from having practically no understanding of this central concept, to virtually the same understanding of it as the teacher. These results corroborate the earlier analysis of the degrees of the concepts within the KSRs. It will be recalled that the students' concepts of sample standard deviation (9), sample variance (10), and variability (14) showed greatest increase in complexity over the course of instruction.

The cluster analysis based on the four central concepts produced much the same results as for the full KSR. Except for some shuffling within the groups, the same distinct pre- and posttest clusters emerged in the dendrogram (Figure 9).

In terms of individual students, the subject who improved the most was Student F, whose *BSED* declined a total of 9, due to his improvement in understanding the ideas of "variability" and "central tendency." His initial, anomalous understanding of the idea of "central tendency" became clear when his KSR vector for this idea was examined: He had included one extraneous connection linking central tendency (1) to range (8) in his structure, and omitted two other relevant connections. This was changed in his posttest KSR when he matched the teacher's KSR for this idea.

Two other students whose representations were of interest were G and H. While their overall similarities to the teacher's KSR increased, on the central concept of "parameter" alone, their similarities decreased.



Teacher's KSR: teach;
Subjects' pretest KSRs: F pre to K pre;
Subjects' posttest KSRs: F post to K post.
Cluster analysis using Average Linkage (Between Groups)

Figure 9. Dendrogram of cluster analysis based on four central concepts.

Conclusion

The lattice-theoretic framework developed here allows for assessments that can compare individual learners' knowledge structure representations to a canonical representation, track changes due to instruction, and diagnose omitted, as well as erroneous, elements in their knowledge.

The definition of knowledge structure representations as elements in a lattice was found to be an effective global characterization. While the literature supports the use of qualitative descriptions of such KSR features as complexity, centrality of concept, and similarity, the lattice-theoretic approach allowed for quantitative measures of cognitive structures to be developed. The changes in students' dissimilarities, ranks, and degrees of the concepts could be examined using the instructor's KSR as a reference point. In addition, the dissimilarities could be analyzed using hierarchical cluster analysis. These analyses were able (a) to detect an increased similarity after instruction of the students' KSR to that of their instructor, (b) to detect changes in the complexity of the students' KSR, (c) to classify and compare subjects' KSR, (d) to examine subjects' understanding of the main ideas in the instructor's KSR and (e) to infer the main ideas of the subjects' KSR and represent them as a composite.

In general, the Binary squared Euclidean provided a straightforward interpretation for the dissimilarity between two knowledge structures representations, as the number of moves that must be made in order to go from one KSR to another in the KSR lattice. The two methods used for analyzing the centrality of a concept in a KSR were essentially complementary: The cluster analysis grouped together those connections to form a composite of what the students understood to be the central concepts, while the method examined the students' KSR in light of the central concepts identified *a priori* in the instructor's KSR. Of these two measurement techniques, comparison to the central concepts in the teacher's KSR proved to be more useful. Although the cluster analysis of the connections was useful in an exploratory sense for determining what the students felt to be the central concepts, comparisons to the instructor's KSR clearly showed how students' corrected their organization of those concepts with instruction. In addition, the technique was able to flag a student's erroneous inclusion a concept that did not belong to a central concept. This ability to track mistakenly added and omitted concepts that for a KSR would be of great use in diagnostic settings.

The two measures of complexity were also complementary. The first method made explicit use of the lattice-theoretic structure in its measure of the rank of a KSR and could be considered as a global measure of complexity. This was a conceptually “clean” way of tying the complexity of a KSR to how “far up” in the KSR lattice it was. The second approach of examining the degree or mean number of connections per concept, could thought of as a local measure of complexity: It allowed us to examine which specific concepts in a student’s KSR were undergoing an increase (or decrease) in its “connectedness” with other concepts. The choice of technique to use depended on whether one needs a either a general, global measure, or a specific, local measure of complexity.[4]

Two steps would seem essential to improving the measures and making them more accessible to a wider audience. First, the lattice-theoretic framework should be extended in such a way as to score the “correctness” of a proposition associated with the connection between a pair of concepts. A scoring system based on the canonical knowledge structure of one or more experts could be used to assess the accuracy of the propositions in a student’s KSR. Such a system would be able to distinguish between propositions that are correct and complete, and those that are valid but show little or no understanding of the relationship

Second, a way to elicit knowledge structure representations needs to be integrated with the means to analyze them. While several computer programs are available for the purpose of eliciting KSRs (e.g., SemNet by Fisher et al., 1988; MicroCAM by Ju, 1989), none of them can automatically compile measures for the characteristics of the representations obtained, and then make comparisons among them. An ambitious design for a program would allow a teacher or researcher to elicit knowledge structure representations from students, and then automatically:

- Compute the rank of each KSR;
- Compute the binary squared Euclidean dissimilarity matrix for all pairwise combinations of KSRs;
- Compute the degree of each concept in a KSR;

[4] Young (1993) extends this analysis by examining the development of students’ knowledge structure representations for different sets of concepts a sequence of statistics courses, and comparing these representations to more than one canonical representation.

- Prepare statistical summaries of rank, dissimilarity, and degree of concept data by group or occasion of testing;

- Compare the characteristics of a group of KSRs with the characteristics of one or more canonical structures;

- Provide graphical output such as the dendrogram of a cluster analysis.

Additional features could include record-keeping functions (such as found in SemNet, which keeps track of relational propositions that have been used) and testing capabilities (such as MicroCam's presentation of a taxonomy of relations for use in a test). The automatization of such features would go a long way in allowing classroom use of the measures presented here.

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